On Some Properties of Middle Cube Graphs and their Spectra

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Abstract: In this research paper we begin with study of family of n dimensional hyper cube graphs and establish some properties related to their distance, spectra, and multiplicities and associated eigen vectors and extend to bipartite double graphs [11]. In a more involved way since no complete characterization was available with experiential results in several inter connection networks on this spectrum our work will add an element to existing theory.

Keywords: Middle cube graphs, Distance-regular graph, Antipodal graph, Bipartite double graph, Extended bipartite double graph, Eigen values, Spectrum, Adjacency matrix.

1. Introduction

An n dimensional hyper cube \( Q_n \) [24] also called n-cube is an “n” dimensional analogue of Square and a Cube. It is stopped up dense arched figure whose 1-skleton comprises of gatherings of inverse equal line fragments adjusted in every one of spaces measurements, opposite to one another and of same length. A factor is a hypercube of dimension zero. If one strikes this factor one-unit length, it will sweep out a line segment, which is the measure polytope of dimension one. If one strikes this line phase its size in a perpendicular course from itself; it sweeps out a two-dimensional square. If one strikes the rectangular one-unit size in the route perpendicular to the plain surface it lies on, it will generate a three-dimensional cube. This can be generalized to any variety of dimensions. For example, if one strikes the dice one-unit size into the fourth dimension, it generates a four-dimensional measure polytope or tesseract.

The number of \( m \)-dimensional hyper cubes on the boundary of an \( n \)-cube is

\[
2^{n-m} \binom{n}{m}
\]

For example, the boundary of a 4-cube contains 8 cubes, 24 squares, 32 lines and 16 vertices.

A unit hyper cube is a hyper cube whose side has length 1 unit whose corners are

\[
2^{n+1}
\]

Points in \( \mathbb{R}^n \) with every organize equivalent to 0 or 1 termed as measure polytope.

The correct number of edges of cube of dimension \( n \) is \( n \times 2^{n-1} \) for example 7-cube has \( 7 \times 2^6 = 448 \) edges.

A. Dimension of the cube

<table>
<thead>
<tr>
<th>No. of vertices</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>No. edges</td>
<td>1</td>
<td>4</td>
<td>12</td>
<td>32</td>
<td>80</td>
<td>192</td>
</tr>
</tbody>
</table>

Here we define adjacency matrix of n cube described in a constructive way.

\[
V_{2^{n+1}} = \begin{pmatrix} V_{2^n} & I_{2^n} \\ I_{2^n} & V_{2^n} \end{pmatrix}
\]
Since \( n \) \( Q_n \) is \( n \) regular bipartite graph of \( 2^n \) vertices characteristic vector of subsets of \([n] = \{1, 2, 3, \ldots, n\}\) vertices of layer \( L_k \) corresponds to subsets of cardinality \( k \).

If \( n \) is odd \( n = 2k - 1 \), the middle two layers \( L_k, L_{k-1} \) of \( Q_n \) with \( nc, nc_{k-1} \) vertices forms middle cube graph \( MQ_k \) by induction.

As \( MQ \) is bipartite double graph which is a sub graph of \( n \) cube \( Q_n \) induced by vertices whose binary representations have either \( k-1 \) or \( k \) no. Of \( 1 \)'s is of \( k \)-regular as shown in figures below.

The middle cube graph \( MQ_3 \) as a subgraph of \( Q_3 \) or as the bipartite double graph of \( O_2 = K_3 \).

\section*{B. Bipartite double graph}

Let \( H = (V, E) \) be a graph of order \( n \), with vertex set \( V = \{1, 2, \ldots, n\} \). Its bipartite double graph \( 1 + \lambda, -1 - \lambda \hat{H} = (V, \hat{E}) \) is the graph with the projected vertex set, \( \hat{V} = \{1, 2, \ldots, n, 1', 2', \ldots, n'\} \) and adjacencies induced from the adjacencies in \( H \) as follows:

\[
\begin{array}{ccc}
\hat{i} & \hat{E} & \hat{j}' \\
\hat{i} & \hat{E} & \hat{j}'
\end{array}
\]

Thus, the edge set of \( \hat{H} \) is \( \hat{E} = \{ij' | ij \in E\} \). From the definition, it follows that \( \hat{H} \).

If \( H \) is a bipartite graph \([24, 21] \) with stable subsets \( V_1 = \{1, 2, \ldots, n\} \), and \( V_2 = \{1', 2', \ldots, n'\} \). For example, if \( H \) is a bipartite graph, then its bipartite double graphs \( \hat{H} \) consists of two non-connected copies of \( H \).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig1.png}
\caption{Path p-4 and its bipartite Double Graph}
\end{figure}

Graph \( H \) has diameter 2 and \( \hat{H} \) has diameter 3

If \( \hat{H} \) is extended bipartite double graph by adding edges \((i, i')\) or each \( i \in V \) \( \hat{H} \equiv \hat{H} \).

\section*{C. Notations}

The cardinality of the graph \( G \) is \( n = |V| \) and its size is \( m = |E| \). Name the vertices using natural numbers \( 1, 2, \ldots, n \). If \( i \) is adjacent to \( j \), that is, \( ij \in E \), we write \( i \parallel j \) or \( i \sim j \). The length between two ends is denoted by \( dist(i, j) \). We also use the concepts of even length and odd length between two ends, denoted by \( dist + \) and \( dist - \), respectively. They are defined as the length of a shortest even, odd walk between the corresponding vertices. The set of vertices which are \( L \)-apart from vertex \( i \), in view of the usual length, is \( \Gamma_i \). The eccentricity of a vertex is \( ecc(i) = \max_{1 \leq l \leq \text{dist}(i, j)} \text{dist}(i, j) \). The diameter of the graph \( G \) is \( D = D(G) = \max_{1 \leq l \leq \text{dist}(i, j)} \text{dist}(i, j) \).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig2.png}
\caption{Graph H has diameter 2 and \( \hat{H} \) has diameter 3}
\end{figure}
The complete bipartite graph $K_{m,n}$ has spectrum $\pm\sqrt{mn}$, $0^{m+n-2}$.

More usually, every bipartite graph has a spectrum that is symmetric w.r.t. the origin: if $\theta$ is eigenvalue, then also $-\theta$, with the same multiplicity.

The n-cube graph (called $2^n$, or $Q_n$) is the chart by means of vertices the binary vectors of length n, where two vectors are adjacent what time they be different in a single position. The 0-cube is $K_1$, the 1-cube is $K_2$, the 2-cube is $C_4$.

The spectrum of $2^n$ consists of the eigenvalues $n-2i$ with multiplicity \[ \binom{n}{i}(0 \leq i \leq n) \]

The absolute bipartite graph $K_{m,n}$ has an adjacency matrix of rank 2, consequently we expect to have eigenvalue 0 of multiplicity $n-2$, and two non-trivial eigenvalues. These should be equal to $\pm \lambda$, because the sum of all eigenvalues is always 0.

We find $\lambda$ by solving $Ax = \lambda x$. By symmetry, we guess that the eigenvector $x$ should have $m$ coordinates equal to $\alpha$ and $n$ coordinates equal to $\beta$. Then,

$$Ax = (m\beta, ..., m\beta, n\alpha, ..., n\alpha)$$

This should be a multiple of $x = (\alpha, ..., \alpha, \beta, ..., \beta)$. Therefore, we get $m\beta = \lambda \alpha$ and $n\beta = \lambda \beta$ i.e. and $mn\beta = \lambda^2 \beta$ and $\lambda = \sqrt{mn}$

A graph $\Gamma$ is called bipartite when its vertex set can be partitioned into two disjoint parts $X_1, X_2$ such that all edges of $\Gamma$ meet both $X_1$ and $X_2$. The adjacency matrix of a bipartite graph has the form $A = \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}$. It follows that the spectrum of a bipartite graph is symmetric w.r.t. 0: if $\begin{bmatrix} u \\ v \end{bmatrix}$ is an eigenvector with eigenvalue 0, then $\begin{bmatrix} u \\ -v \end{bmatrix}$ is also an eigenvector with eigenvalue 0.

For the ranks one has $\text{rk} A = 2 \text{ rk } B$. If $n_i = |X_i|$ (i = 1, 2) and $n1 \geq n2$, then $\text{rk} A \leq 2n2$, so that $\Gamma$ has eigenvalue 0 with multiplicity at least $n1 - n2$.

One cannot, in general, recognize bipartiteness from the Laplace or signless Laplace spectrum. E.g., $K_{1,3}$ and $K_4 + K_4$ have the same signless Laplace spectrum and only the former is bipartite.
However, by Proposition below, a graph is bipartite precisely when its Laplace spectrum and signless Laplace spectrum coincide.

A. Elementary Graphs associated Eigenvalues

<table>
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<tr>
<th>label</th>
<th>picture</th>
<th>A</th>
<th>L</th>
<th>Q</th>
<th>R</th>
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<td></td>
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<td>1</td>
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</table>

B. Characteristic polynomial

Let \( \Gamma \) be a directed graph on \( n \) vertices. For any directed subgraph \( C \) of \( \Gamma \) that is a union of directed cycles, let \( c(C) \) be its figure of cycles. Then the total polynomial \( p(A) = \det(tI - A) \) of \( \Gamma \) can be expanded as \( \sum C_i t^{n-i} \) where \( C_i = \sum C(-1)^e(c(C)) \) with \( C \) association over all usual heading for subgraphs with in- and outdegree 1 in i vertices. (Indeed, this is just a reformulation of the definition of the determinant as, \( \det M = \sum_\sigma \sgn(\sigma) M_{1\sigma(1)} \cdots M_{n\sigma(n)} \) Note that when the variation \( \sigma \) with \( n-i \) fixed points is written as a product of non-identity cycles, its sign is \( (-1)^e \) where \( e \) is the number of even cycles in this product. Since the number of odd non-identity cycles is congruent to \( i \mod 2 \), we have \( \sgn(\sigma) = (-1)^{i+e(\sigma)} \).

For example, the triangle has \( c_0 = 1, c_2 = -1 \). Directed edges that do not occur in directed cycles do not influence the (ordinary) spectrum. The same description of \( p_A(i) \) holds for undirected graphs (with each edge viewed as a pair of opposite directed edges).

Since \( \frac{d}{dt} \det(tI - A) = \sum \det(tI - A) \) where \( A_\sigma \) is the submatrix of \( A \) obtained by deleting row and column \( x \), it follows that \( \sigma^\prime, A(i) \) is the sum of the characteristic polynomials of all single-vertex-deleted sub-graphs of \( \Gamma \). The spectrum of the complete bipartite graph \( K_{m,n} \) is \( \pm \sqrt{mn}, 0^{m+n-2} \). The Laplace spectrum is \( 0^l, m^{n-1}, m^{m-1}, (m+n)^l \).

The main eigenvalue of a graph is also recognized as its spectral radius or index. The basic in sequence concerning the main eigenvalue of a (possibly directed) graph is provide by Perron-Frobenius theory as follows.

C. Proposition

Each graph \( \Gamma \) has a real eigenvalue \( \theta_0 \) with positive real corresponding eigen-vector, and such that for each eigenvalue \( \theta \) we have \( |\theta| \leq \theta_0 \).

The value \( \theta_0 (\Gamma) \) remains same when vertices or edges are removed from \( \Gamma \). Let \( \Gamma \) is strongly connected. Then

(i) \( \theta_0 \) has multiplicity 1.

(ii) If \( \Gamma \) is primitive (strongly connected, and such that no cycles have a length equal to multiple of some integer \( d > 1 \), then \( |\theta| < \theta_0 \) for all eigen-values \( \theta \) different from \( \theta_0 \).

(iii) The value \( \theta_0 (\Gamma) \) reduces when vertices or edges are removed from \( \Gamma \). Now let \( \Gamma \) be undirected. By Perron-Frobenius theory and interlacing we find an upper and lower bound for the maximum characteristic value of a connected graph.

(Note that \( A \) is irreducible if and only if \( \Gamma \) is connected.) Out of all the connected graphs \( \Gamma \), with non-primitive \( A \) are precisely the bipartite diagram (with period 2) is explained in the following proposition.

D. Proposition

(i) A graph \( \Gamma \) is bipartite if and only if for each eigenvalue \( \theta \) of \( \Gamma \), also \(-\theta \) is an eigen-value, with the same multiplicity.

(ii) If \( \Gamma \) is connected with largest eigen-value \( \theta_1 \), then \( \Gamma \) is bipartite if and only if \(-\theta_1 \) is an eigen-value of \( \Gamma \).

Proof. For connected graphs all is clear from the Perron-Frobenius theorem. That gives (ii) and (by taking unions) the ‘only if’ part of (i). For the ‘if’ part of (i), let \( \theta_1 \) be the spectral radius of \( \Gamma \). Then some connected component of \( \Gamma \) has eigenvalues \( \theta_1 \) and \(-\theta_1 \), and hence is bipartite. Removing its contribution to the spectrum of \( \Gamma \), we see by acceptance on the quantity of parts that all Parts are bipartite.

We build up some more hypotheses stretched out on spectra and multiplicities and related eigen values which are reached out to bipartite twofold charts.

Theorem: Let \( F \) be a field, let \( R \) be a commutative sub ring to \( F^{n \times n} \), the set of all n * n matrices over F. Let \( M \in R^{m \times n} \), then

\[
\det_F(M) = \det_F(\det_F(M))
\]

\[
\vdots \quad \det_F^k(M) = \det_F(AD - BC).
\]
for a bipartite twice chart feature polynomial. [13]
We show the next theorems presentation geometric
multiplicities of eigen value \( \lambda \) of \( H \) \( \Rightarrow \) geometric
multiplicities of eigen values \( \lambda \) and \(-\lambda\) of \( \hat{H} \)
\[ 1 + \lambda, -1 - \lambda \text{ of } \hat{H} \]

**Theorem:** Let \( H \) be a graph on \( n \) vertices, with the adjacency
matrix \( A \) and characteristic,
\[ \bar{u} = u^*_j v_j (1 + \lambda) \perp \]
\[ (u^*_j) = \sum_{j \in i^*} v_j = \lambda v_i = \lambda u_i^* \]

polynomial \( \mathcal{O}_H(x) \). Then, the characteristic polynomials of
\( \hat{H} \) and \( H \) are, respectively,
\[ \mathcal{O}_H(x) = (-1)^n \mathcal{O}_H(x) \mathcal{O}_H(-x), \]
\[ \mathcal{O}_H(x) = (-1)^n \mathcal{O}_H(x-1) \mathcal{O}_H(-x-1). \]

Adjacency matrices are, in that order,
\[ A = \begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix} \text{ and } \hat{A} = \begin{pmatrix} 0 & A+1 \\ A+1 & 0 \end{pmatrix}. \]

With the above corollary
\[ \mathcal{O}_H(x) = \det(xI_n - \hat{A}) = \det \left( \begin{array}{cc} xI_n & -A \\ -A & xI_n \end{array} \right) = \det(x I_n - A) \det(xI + A) = (..1) (x) (-x), \]

Whereas the characteristic polynomial of \( \hat{H} \) is
\[ \mathcal{O}_H(x) = \det(xI_n - \hat{A}) = \det \left( \begin{array}{cc} xI_n & -A \\ -A & xI_n \end{array} \right) \\
= \det(x I_n - A) \det(xI + A) = (-1)^n \mathcal{O}_H(x-1) \mathcal{O}_H(-x-1). \]

**Theorem:** Let \( H \) be a graph and \( v \) a \( \lambda \)-eigenvector \( H \). Let us
consider the vector \( u^+ \) with Components \( u_i^+ = v_i \), \( u^- \) with
components \( u_i^- = -v_i \) for \( 1 \leq i, i' \leq n \)

Then,
\[ u^+ \lambda \text{-eigenvector } \hat{H} \text{ and } (1 + \lambda) \text{ eigenvector } \hat{H} \]
\[ \bar{u} - \lambda \text{-eigenvector } \hat{H} \text{ and } (-1 - \lambda) \text{ eigenvector } \hat{H} \]

If the vertex \( i, 1 \leq i \leq n \), all its nearby vertices are of type \( j^* \),
with \( i \in (E) \cup (j^*) \). Then
\[ \sum_{j \in (i^*)} u^+_j = \sum_{j \in i^*} v_j = \lambda v_i = \lambda u^+_i \]

If the vertex \( i', 1 \leq i' \leq n \), all its neighboring vertices are of
type \( j \), with \( i \in (E) \cup (j) \).

Then
\[ (Au^+) = \sum_{j \in (i^*)} u^+_j = \sum_{j \in i^*} v_j = \lambda v_i = \lambda u^+_i \]

By a comparable reasoning with \( u^- \), we obtain
\[ (Au^-)_i = \sum_{j \in (i^*)} u^-_j = \sum_{j \in i^*} v_j = -\lambda u^-_i \]

Also \( 1 + \lambda, -1 - \lambda \) are eigen values for \( u^+ \), \( u^- \) eigen vectors of \( \hat{H} \).

From the beyond information realizing an isomorphism [8, 2] defined by
\[ f : V[\tilde{\Omega}_k] \to V[\Omega_k], \]
\[ u \mapsto u \]
\[ u^+ \mapsto \bar{u} \]

is a surjection by the definition of bipartite double graph, if \( u \) and \( v' \) are two vertices of \( \tilde{\Omega}_k \).

The center cube graph \( \Omega_k \) with \( D=2k-1 \) by above corollary
is isomorphic to \( \tilde{\Omega}_k \).

We authenticate spectrum of \( Q_{2k-1} \) hold all eigen values of \( [\Omega_k] \),
\[ \theta^+_i = (-1)^i (k-i) \text{ and } \theta^-_i = -\theta^+_i \text{ for } 0 \leq i \leq k - 1 \]

With multiplicities \( m(\theta^+_i) = m(\theta^-_i) = \frac{k-1}{k} \left( \begin{array}{c} 2k \end{array} \right) \)

3. Conclusion

In Verification of the above results,
\[ spMQ_3 = \{ \pm 2, \pm 1^2 \} \]
\[ spMQ_5 = \{ \pm 3, \pm 2^4, \pm 1^5 \} \]
\[ spMQ_7 = \{ \pm 4, \pm 3^6, \pm 2^{14}, \pm 1^{14} \} \]
\[ spMQ_9 = \{ \pm 5, \pm 4^8, \pm 3^{27}, \pm 2^{48}, \pm 1^{42} \} \]

For uppermost degree Distance polynomials of \( [\Omega_k] \)
p5 (3) = p5 (1) = p5 (-1) = 1 and p5 (2) = p5 (-2) = p5 (-3) = -1. Then,
\[ m(\lambda_0) = m(\lambda_4) = m(\lambda_5) = 1, \]
\[ m(\lambda_1) = m(\lambda_4) = m(\lambda_5) = 4, \]
\[ m(\lambda_2) = m(\lambda_1) = m(\lambda_2^2) = 5, \]

**References**


