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Harmonic Mean Topological Indices of Graphs

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***Abstract*: In Quantitative structure–activity relationship (QSAR) and Qualitative structure-property relationship (QSPR) analysis, chemical graph theory plays an indispensable role. The physico-chemical properties of molecules can be studied by using the information encoded in their corresponding chemical (molecular) graphs. Graph-theoretical invariants of graphs are called topological indices or molecular descriptors. They are numerical values associated with chemical graphs in path molecular graphs. A graph invariant is any function on a graph that does not depends on labeling of its vertices. A large number of different invariants, have been employed with various degrees of success in QSAR and QSPR. Here we introduce a new topological indices of a graph G named as harmonic mean topological indices denoted by $H_{MI}(G)$ and determine its values for some standard graphs, some special graphs and their line graphs. In addition, harmonic mean indices of certain graph operations and some bounds for them are obtained.**

***Keywords*: Topological indices, Harmonic mean indices, Line graphs, Graph operations.**

1. Introduction

A graph $G=(V,E)$ is an ordered pair where V is a non empty set and E is a set of unordered pairs of elements of V. Elements of V are called the vertices and the set is known as a vertex set of G denoted by V(G). Similarly, elements of E are called edges of G and the set is called as edge set of G denoted by E(G). The cardinality of $V(G)$ is called the order of G and that of $E(G)$ is called the size of the graph. The degree of a vertex $v \in V(G)$ is denoted by $d_G(v)$ or simply $d(v)$. [12] The degree $d_G(e)$ of an edge $e=uv$ of G is given by $d_G(e) = d_G(u)+d_G(v)-2$. All graphs under our consideration are finite, simple and undirected.

The topological indices play a vital role in chemical documentation, isomer discrimination, relationship analysis like QSAR and QSPR.

In 1947, Weiner used his [13] topological index named Weiner index to calculate the boiling point of paraffins. Then in 1972 Gutman and Trinajstic defined the Zagreb indices [6], [18] which are popular. There after many indices are defined named Randic index [4], ABC indices [18], RDD indices [12], Hyper Zagreb indices [10], forgotten topological indices [12], geometric-arithmetic index [11], harmonic indices [16], Kulli-Basava indices [4], connectivity index etc. Here we introduce a new parameter known as harmonic mean index denoted by $H_{MI}(G)$ and defined as $H_{MI}(G) = \sum_{uv\in E(G)} \frac{2du.dv}{du+dv}$ where du and dv represent the degrees of the vertices with which the edge uv of the graph G incident with.

2. Harmonic mean indices of some family of graphs

In this section we obtain an explicit formula for harmonic mean indices of some family of graphs namely $P_n, C_n, K_n, K_{m,n}, W_n, S_n, G_n, f_n$. [5], [3] and their line graphs and some relations connecting them. The line graph of a graph G denoted as $L(G)$ and is obtained from G by considering the edges of G as vertices of L(G) and two vertices are adjacent in L(G) if the corresponding edges are adjacent in G.

Definition 2.1. The harmonic mean indices of a nontrivial graph G denoted by $H_{MI}(G)$is defined as $H_{MI}(G) = \sum_{uv\in E(G)} \frac{2du.dv}{du+dv}$ where du, dv represent the degrees of the vertices of the edge uv

Theorem 2.2. For the path graph P_n, the harmonic mean index is given by

(a) $$H_{MI}(P_n) = \begin{cases} 1 & for\ n=2 \\ \frac{8}{3}+2(n-3) & for\ n\geq 3 \end{cases}$$

(b) $$H_{M1}(L(P_n)) = \begin{cases} 1 & for\ n=3 \\ \frac{8}{3}+2(n-4) & for\ n\geq 4 \end{cases}$$

Proof. (a) $P_n,\ n\geq 3$ has $n-3$ edges with end vertex degrees 2 each and two edges with end vertex degrees 1 and 2. Then

$$H_{MI}(P_n) = \sum_{uv\in E((P_n)} \frac{2du.dv}{du+dv} = \frac{2\times2\times1}{2+1}\times 2 + \frac{2\times2\times2}{2+2}(n-3)$$
$$= \frac{8}{3}+2(n-3)$$

Also for $n=2$, there is only one edge and two pendant vertices, gives

$$H_{MI}(P_n) = 1 \quad for\ n=2$$

(b) The line graph of P_n that is $L(P_n)$ is nothing but P_{n-1} and hence the result for $H_{MI}(L(P_n))$ follows from(a). □

Corollory 2.3. $H_{MI}(L(P_n)) = H_{MI}(P_n) - 2\ for\ n\geq 4$

Theorem 2.4. For a complete graph K_n, the harmonic mean index is given by

(a) $H_{MI}(K_n) = \frac{n(n-1)^2}{2}$

(b) $H_{MI}(L(K_n)) = n(n-1)(n-2)^2$

Proof. (a) K_n has $\binom{n}{2}$ edges and degree of each vertex is n-1.

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$$H_{MI}(K_n) = \sum_{uv\in E((P_n)} \frac{2d_{K_n}(u)d_{K_n}(dv)}{d_{K_n}u+d_{K_n}v}$$
$$= \sum_{uv\in E((P_n)} \frac{2(n-1)(n-1)}{n-1+n-1}$$
$$= (n-1)\sum_{uv\in E((P_n)} 1$$
$$= (n-1)\binom{n}{2}$$
$$= \frac{n(n-1)^2}{2}$$

(b) The line graph $L(K_n)$ has $\binom{n}{2}$ vertices and $2(n-2)$ regular so that there are $\binom{n}{2}(n-2)$ edges. Then, $H_{MI}(L(K_n)) =$

$$\sum_{uv\in E(K_n)} \frac{2d_{L(K_n)}(u)d_{L(K_n)}(v)}{d_{L(K_n)}u+d_{L(K_n)}(v)}$$
$$= \sum_{uv\in E(K_n)} \frac{2.2(n-2).2(n-2)}{2(n-2)+2(n-2)}$$
$$= 2(n-2)\sum_{uv\in E(K_n)} 1$$
$$= 2(n-2)\frac{n(n-1)(n-2)}{2}$$
$$= n(n-1)(n-2)^2$$

Observation 2.5. $H_{MI}(L(K_n)) = 2nH_{MI}(K_{n-1})$

Theorem 2.6. For a complete bipartite graph $K_{m,n}$ and for its line graph the harmonic mean index are given by

(a) $H_{MI}(K_{m,n}) = \frac{2(mn)^2}{m+n}$

(b) $H_{MI}(L(K_{m,n})) = \frac{mn(m+n-2)^2}{2}$

Proof. (a) Each edge of $K_{m,n}$ has end vertices of degree m and n and there are mn such edges.

$$H_{MI}(K_{m,n}) = \sum_{uv\in E(K_{m,n})} \frac{2d_{K_{m,n}}(u)d_{K_{m,n}}(dv)}{d_{K_{m,n}}u+d_{K_{m,n}}v}$$
$$= \sum_{uv\in E((K_{m,n})} \frac{2mn}{n+m}$$
$$= \frac{2mn}{n+m}\sum_{uv\in E((K_{m,n})} 1$$
$$= \frac{2mn}{n+m}\text{mn} = \frac{2(mn)^2}{m+n}$$

(b) $L(K_{m,n})$ has mn vertices each of degree $m+n-2$. Then number of edges of $L(K_{m,n})$ has degree $\frac{mn(m+n-2)}{2}$ and then by case (a) the result will be obtained.

$$H_{MI}(L(K_{m,n})) =$$
$$= \sum_{uv\in E(L(K_{m,n}))} \frac{2(m+n-2)(m+n-2)}{m+n-2+m+n-2}$$
$$= (m+n-2)\sum_{uv\in E(L(K_{m,n}))} 1$$
$$= (m+n-2)\frac{mn(m+n-2)}{2}$$
$$= \frac{mn(m+n-2)^2}{2}$$

Corollory 2.7. For the star $S_n = K_{1,n}$,

$$H_{MI}(K_{1,n}) = \frac{2n^2}{n+1} \text{ and}$$
$$H_{MI}(L(S_n)) = H_{MI}(L(K_{1,n})) = \frac{n(n-1)^2}{2}$$

Corollory 2.8. $H_{MI}(L(S_n)) = \frac{n^2}{4} H_{MI}(S_{n-1})$

Proposition 2.9. $H_{MI}(C_n) = H_{MI}(L(C_n)) = 2n$

Proof. Since C_n is 2 regular and has n edges $H_{MI}(C_n) = 2n$.

From the fact that $L(C_n)$ is isomorphic to C_n, completes the proof. □

Theorem 2.10. For the wheel graph W_n with $n+1$ vertices, the harmonic mean index is given by

$$H_{MI}(W_n) = \frac{9n(n+1)}{n+3}$$

Proof. Let $V(W_n) = V_1 \cup V_2$ and $E(W_n) = E_1 \cup E_2$ where

$V_1 = \{v_c | v_c \text{ isthe centre vertex of } W_n$ with degree $n\}$

$V_2 = V(W_n) \setminus \{v_c\} = \{v_i, \text{i} = 1,2,\dots,\text{n}\}$

And $E_1 = \{v_c v_j \in E(W_n), v_c \in V_1 \text{and } v_i \in V_2\}$, $|E_1| = n$

$E_2 = \{v_i v_j \in E(W_n);\ v_i, v_j, \in V_2\}$, $|E_2| = n$

For $v_i v_j \in E_1$, $d_{W_n}(v_c) = n$ and $d_{W_n}(v_i) = 3$ for $i = 1,\dots,n$ and for

$v_i v_j \in E_2$, $d_{W_n}(v_i) = 3 \text{ for } i = 1,\dots,n$.

Hence $H_{MI}(W_n) = \sum_{uv\in E_1} \frac{2d_{W_n}(u).d_{W_n}(v)}{d_{W_n}(u)+d_{W_n}(v)} + \sum_{uv\in E_2} \frac{2d_{W_n}(u).d_{W_n}(v)}{d_{W_n}(u)+d_{W_n}(v)}$

$$= \sum_{uv\in E_1} \frac{2.3.n}{3+n} + \sum_{uv\in E_2} \frac{2.3.3}{3+3}$$
$$= \frac{6n}{3+n}\sum_{uv\in E_1} 1 + 3\sum_{uv\in E_2} 1$$
$$= \frac{6n^2}{3+n} + 3n$$
$$= \frac{9n^2+9n}{n+3} = \frac{9n(n+1)}{n+3}$$ □

Theorem 2.11. For the wheel graph W_n with $n+1$ vertices, the harmonic mean index is given by

$$H_{MI}(L(W_n)) = \frac{n(n^2-1)}{2} + \frac{16n(n+1)}{n+5} + 4n$$

Proof. For the structure $L(W_n)$ we have $|V(L(W_n))| = 2n$ and $|E(L(W_n))| = \frac{n^2+5n}{2}$

$V(L(W_n)) = V_1 \cup V_2$ and $E(L(W_n)) = E_1 \cup E_2 \cup E_3$ where,

$V_1 = \{v_i \in V(L(W_n))/d_{L(W_n)}(v_i) = n+1,\ i = 1,2,\dots,n\}$,

$V_2 = \{v_i \in V(L(W_n))/d_{L(W_n)}(v_i) = 4,\ i = n+1,\dots,2n\}$,

$E_1 = \{e_{v_i v_j} \in E(L(W_n));\ v_i, v_j, \in V_1\}$

$E_2 = \{e_{v_i v_j} \in E(L(W_n));\ v_i, v_j, \in V_2\}$

$E_3 = \{e_{v_i v_j} \in E(L(W_n));\ v_i \in V_1, v_j \in V_2\}$

so that $|E_1| = \frac{n(n-1)}{2}$, $|E_2| = n$, $|E_3| = 2n$, then

$$H_{MI}(L(W_n)) = \sum_{uv\in E_1} \frac{2d_{L(W_n)}(u).d_{L(W_n)}(v)}{d_{L(W_n)}(u)+d_{L(W_n)}(v)} + \sum_{uv\in E_2} \frac{2d_{L(W_n)}(u).d_{L(W_n)}(v)}{d_{L(W_n)}(u)+d_{L(W_n)}(v)} + \sum_{uv\in E_3} \frac{2d_{L(W_n)}(u).d_{L(W_n)}(v)}{d_{L(W_n)}(u)+d_{L(W_n)}(v)}$$

$$= \sum_{uv\in E_1} \frac{2(n+1)(n+1)}{2(n+1)} + \sum_{uv\in E_2} \frac{2.4.4}{4+4} + \sum_{uv\in E_3} \frac{2(n+1).4}{n+5}$$

$$= (n+1)|E_1| + 4|E_2| + \frac{8(n+1)}{n+5}|E_3|$$
$$= \frac{n(n^2-1)}{2} + 4n + \frac{16n(n+1)}{n+5}$$

Definition [3]A gear graph G_n of order $2n+1$ is a wheel graph with a vertex added between each pair of adjacent vertices of the outer cycle. ie, it includes an even cycle C_{2n} and the vertices of C_{2n} in G_n are of two kinds, say vertices of degree 2 and vertices of degree 3. The vertices of degree 3 are called major vertices and that of 2 are called minor vertices. The central vertex of G_n denoted by v_0 has degree n. Note that it has $3n$ edges.

Theorem 2.12 The harmonic mean index $H_{MI}(G_n) = = \frac{18n(3n+4)}{5(n+3)}$ where G_n is the gear graph of order $2n+1$.

Proof.

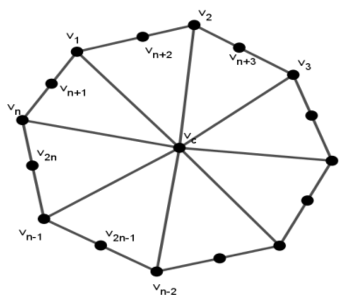

Fig. 1. G_n

The vertex set of gear graph can be partitioned as follows.

$$V(G_n) = V_1 \cup V_2 \cup V_3 \quad \text{and} \quad E(W_n) = E_1 \cup E_2$$

where

$V_1 = \{v_0 | v_0 \in V(G_n) \text{ and } d_{G_n}(v_c) = n\}$, $|V_1| = 1$

$V_2 = \{v_i | v_i \in V(G_n) \text{ and } d_{G_n}(v_i) = 3, i = 1,2,\dots,n\}$, $|V_2| = n$

$V_3 = \{v_i | v_i \in V(G_n) \text{ and } d_{G_n}(v_i) = 2, i = 1,2,\dots,n\}$, $|V_1| = n$

And $E_1 = \{v_0 v_i \in E(G_n), v_0 \in V_1 and\ v_i \in V_2\}$, $|E_1| = n$,

$E_2 = \{v_i v_j \in E(G_n);\ v_i, v_j, \in V_2\}$, $|E_2| = 2n$

For $v_i v_0 \in E_1$, $d_{W_n}(v_0) = n$ and $d_{W_n}(v_i) = 3\ for\ i = 1,\dots,n$ and for $v_i v_j \in E_2, d_{W_n}(v_i) = 3$ $d_{W_n}(v_i) = 2$, for $for\ i,j = 1,\dots,2n$.

Now $H_{MI}(G_n) = \sum_{uv\in E_1} \frac{2d_{G_n}(u).d_{G_n}(v)}{d_{G_n}(u)+d_{G_n}(v)} + \sum_{uv\in E_2} \frac{2d_{G_n}(u).d_{G_n}(v)}{d_{G_n}(u)+d_{G_n}(v)}$

$$= \sum_{uv\in E_1} \frac{2.n.3}{n+3} + \sum_{uv\in E_2} \frac{2.3.2}{3+2}$$
$$= \frac{6n}{n+3} \sum_{uv\in E_1} 1 + \frac{12}{5}.\sum_{uv\in E_2} 1$$
$$= \frac{6n^2}{n+3} + \frac{24n}{5}$$
$$= \frac{18n(3n+4)}{5(n+3)}$$

Theorem 2.13. For the line graph of the gear graph G_n of order $2n+1$, the harmonic mean index is given by $H_{MI}(L(G_n)) = \frac{n(n+1)(n^2+3n+20)}{2(n+4)} + 6n$

Proof. Let the vertex and edge set of $L(G_n)$ be partitioned as $V(L(G_n)) = V_1 \cup V_2$ and $E(L(G_n)) = E_1 \cup E_2$.Then $|V(L(G_n))| = 3n$ and $|E(L(G_n))| = \frac{n^2+7n}{2}$.

Moreover,

$V_1 = \{v_i \in V(L(G_n)) | d_{L(G_n)}(v_i) = n+1, i = 1,..n\}$ and

$V_2 = \{v_i \in V(L(G_n)) | d_{L(G_n)}(v_i) = 3, i = n+1,\dots,3n\}$.

Furthermore, the sets E_1, E_2, E_3 are

$$E_1 = \{v_i v_j \in E(L(G_n));\ v_i, v_j \in V_1\}$$
$$E_2 = \{v_i v_j \in E(L(G_n));\ v_i, v_j \in V_2\}$$
$$E_3 = \{v_i v_j \in E(L(G_n));\ v_i \in V_1, v_j \in V_2\}$$

so that $|E_1| = \binom{n}{2}$, $|E_2| = 2n$, $|E_3| = 2n$, then

$$H_{MI}(L(G_n)) = \sum_{uv\in E_1} \frac{2d_{L(G_n)}(u).d_{L(G_n)}(v)}{d_{L(G_n)}(u)+d_{L(G_n)}(v)} + \sum_{uv\in E_2} \frac{2d_{L(G_n)}(u).d_{L(G_n)}(v)}{d_{L(G_n)}(u)+d_{L(G_n)}(v)} + \sum_{uv\in E_3} \frac{2d_{L(G_n)}(u).d_{L(G_n)}(v)}{d_{L(G_n)}(u)+d_{L(G_n)}(v)}$$
$$= \sum_{uv\in E_1} \frac{2(n+1)(n+1)}{2(n+1)} + \sum_{uv\in E_2} \frac{2.3.3}{3+.3} + \sum_{uv\in E_3} \frac{2(n+1).3}{3+n+1}$$
$$= (n+1)\sum_{uv\in E_1} 1 + 3\sum_{uv\in E_2} 1 + \frac{6(n+1)}{n+4}\sum_{uv\in E_1} 1$$
$$= \frac{(n+1)n(n-1)}{2} + 3.2n + \frac{6(n+1)}{n+4}.2n$$
$$= \frac{n(n+1)(n^2+3n+20)}{2(n+4)} + 6n$$

Definition.[3] A friendship graph f_n is a collection of n triangles with a common vertex. ie, $f_n = K_1 + nK_2$. It can be obtained from a wheel W_{2n} with a single cycle C_{2n} by deleting alternate edges of the cycle. Let v_c denotes the central vertex, then $d_{f_n}(v_c) = 2n$. Note that f_n has $2n+1$ vertices and $3n$ edges.

Theorem 2.14. For the friendship graph f_n of order $2n+1$, the harmonic mean index is given by

$$H_{MI}(f_n) = \frac{2n(5n+1)}{n+1}$$

Proof.

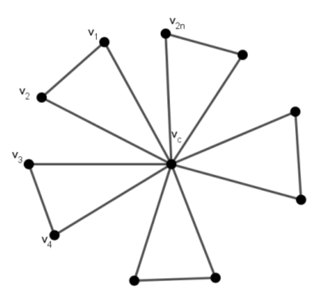

Fig. 2. f_5

The edges of f_n can be partitioned into two types as follows

$E_1 = \{uv \in E(f_n) | d_{f_n}(u) = d_{f_n}(v) = 2\}$ and $|E_1| = n$

$E_2 = \{uv \in E(f_n) | d_{f_n}(u) = 2, \; d_{f_n}(v) = 2n\}$, $|E_2| = 2n$

$$H_{MI}(f_n) = \sum_{uv\in E_1} \frac{2d_{f_n}(u).d_{f_n}(v)}{d_{f_n}(u)+d_{f_n}(v)} + \sum_{uv\in E_2} \frac{2d_{f_n}(u).d_{f_n}(v)}{d_{f_n}(u)+d_{f_n}(v)}$$
$$= \sum_{uv\in E_1} \frac{2.2.2}{2+2} + \sum_{uv\in E_2} \frac{2.2.2n}{2+2n}$$
$$= 2.n + \frac{4n}{1+n}.2n$$
$$= \frac{2n(5n+1)}{n+1}$$

Theorem 2.15. For the friendship graph f_n of order $2n+1$, the harmonic mean index is given by

$$H_{MI}(L(f_n)) = \frac{2n^2(2n^2+n+3)}{n+1}$$

Proof. We have $|V(L(f_n))| = 3n$ *and* $|E(L(f_n))| = 2n^2 + n$.

The vertex set and edge set of $L(f_n)$ can be partitioned as follows.

$V_1 = \{v_i \in V(L(f_n)) | d_{L(f_n)}(v_i) = 2n, i = 1,2,\dots,n\}$ and

$V_2 = \{v_i \in V(L(f_n)) | d_{L(f_n)}(v_i) = 2, i = 2n+1,\dots,3n\}$.

Furthermore, $E_1 = \{v_iv_j \in E(L(f_n));\; v_i, v_j \in V_1\}$ and $E_2 = \{v_iv_j \in E(L(f_n));\; v_i \in V_1, v_j \in V_2\}$

so that $|E_1| = \binom{2n}{2}$, $|E_2| = 2n$,

then
$$H_{MI}(L(f_n)) = \sum_{uv\in E_1} \frac{2d_{L(f_n)}(u).d_{L(f_n)}(v)}{d_{L(f_n)}(u)+d_{L(f_n)}(v)} + \sum_{uv\in E_2} \frac{2d_{L(f_n)}(u).d_{L(f_n)}(v)}{d_{L(f_n)}(u)+d_{L(f_n)}(v)}$$
$$= \sum_{uv\in E_1} \frac{2.2n.2n}{2n+2n} + \sum_{uv\in E_2} \frac{2.2n.2}{2n+2}$$
$$= 2n\sum_{uv\in E_1} 1 + \frac{4n}{n+1}\sum_{uv\in E_2} 1$$
$$= \frac{2n.2n(2n-1)}{2} + \frac{4n}{n+1}.2n$$
$$= 2n^2(2n-1) + \frac{8n^2}{n+1}$$
$$= \frac{2n^2(2n^2+n+3)}{n+1}$$

3. Harmonic mean indices of Cartesian product of some graphs

Graph operations play a major role because some important graphs can be obtained by some graph operations. Composition, Cartesian product, corona product, join, tensor products etc. give some special types or family of graphs. In this section we discuss about the harmonic mean indices of some Cartesian product of graphs like ladder graph, nanotubes.

Definition 3.1 [7],[5],[2] The Cartesian product of two graphs G_1 and G_2 denoted by $G_1 \times G_2$ is a graph with vertex set $V(G_1) \times V(G_2)$ and (u_i, v_j) , (u_k, v_l) are adjacent in $G_1 \times G_2$ if and only if $u_i = u_k$ and $v_jv_l \in E(G_2)$ or $u_iu_k \in E(G_1)$ *and* $v_j = v_l$.

Definition 3.2 [17],[2] The ladder graph L_n is the Cartesian product of P_2 and P_n, that is $P_2 \times P_n$.

Theorem 3.3 The harmonic mean index of the ladder graph is given by

$$H_{MI}(L_n) = \begin{cases} 9n - \frac{52}{5} & for\ n \geq 3 \\ 8 & for\ n = 2 \end{cases}$$

Proof: .Clearly, for $= 2$,$H_{MI}(P_2 \times P_2)$ =8.

The vertex set $V(L_n)$ and edge set $E(L_n)$ can be partitioned as follows.

$V(L_n) = V_1 \cup V_2$ and $E(L_n) = E_1 \cup E_2 \cup E_3$.

Then $|V(L_n)| = 2n$ and $|E(L_n)| = 3n - 2$.

Moreover,

$V_1 = \{v_i \in V(L_n) | d(L_n)(v_i) = 2\}$ and

$V_2 = \{v_i \in |V(L_n) | d_{L_n}(v_i) = 3\}$.

Furthermore, the sets E_1, E_2, E_3 are

$E_1 = \{v_iv_j \in E(L_n);\; v_i, v_j \in V_1\}$,

$E_2 = \{v_iv_j \in E(L_n);\; v_i, v_j \in V_2\}$

$E_3 = \{v_iv_j \in E(L_n);\; v_i \in V_1, v_j \in V_2\}$.

Note that $|E_1|, |E_2|, |E_3|$ are $2,\; 3n-8,\; 4$ respectively for $n \geq 3$.

$$H_{MI}(L_n) = \sum_{uv\in E_1} \frac{2d_{L_n}(u).d_{L_n}(v)}{d_{L_n}(u)+d_{L_n}(v)} + \sum_{uv\in E_2} \frac{2d_{L_n}(u).d_{L_n}(v)}{d_{L_n}(u)+d_{L_n}(v)} + \sum_{uv\in E_3} \frac{2d_{L_n}(u).d_{L_n}(v)}{d_{L_n}(u)+d_{L_n}(v)}$$
$$= 2\sum_{uv\in E_1} 1 + 3.\sum_{uv\in E_2} 1 + \frac{2.2.3}{5}\sum_{uv\in E_3} 1$$
$$= 2.2 + 3(3n-8) + \frac{2.2.3}{5}.4$$
$$= 9n - 20 + \frac{48}{5} = 9n - \frac{52}{5}$$

Definition 3.4. [1], [4], [12] The C_4 -nanotube $TUC_4(m,n)$ is the Cartesian product of P_n *and* C_n , that is $P_n \times C_m$

Theorem 3.5. The harmonic mean index of the graph$P_n \times C_m$ is given by

$$H_{MI}(P_n \times C_m) = 6m + \frac{48\,m}{7} + 4\text{m}(2n-1)$$

Proof. There are mn vertices for $P_n \times C_m$. The edge set $E(P_n \times C_m)$ can be partitioned as follows.

$V_1 = \{v_i \in V(P_n \times C_m) | d_{P_n\times C_m}(v_i) = 3\}$

$V_2 = \{v_i \in V(P_n \times C_m) | d_{P_n\times C_m}(v_i) = 4\}$

$E_1 = \{e_{v_iv_j} \in E(P_n \times C_m);\; v_i, v_j \in V_1\}$,$|E_1| = 2m$

$E_2 = \{e_{v_iv_j} \in E(P_n \times C_m);\; v_i, v_j \in V_2\}$,$|E_2| = 2m$

$E_3 = \{e_{v_iv_j} \in E(P_n \times C_m);\; v_i \in V_1, v_j \in V_2\}$,

$|E_3| = (2n-1)m$

$$H_{MI}(P_n \times C_m) = \sum_{uv\in E_1} \frac{2d_{P_n\times C_m}(u).d_{P_n\times C_m}(v)}{d_{P_n\times C_m}(u)+d_{P_n\times C_m}(v)} + \sum_{uv\in E_2} \frac{2d_{P_n\times C_m}(u).d_{P_n\times C_m}(v)}{d_{P_n\times C_m}(u)+d_{P_n\times C_m}(v)} + \sum_{uv\in E_3} \frac{2d_{P_n\times C_m}(u).d_{P_n\times C_m}(v)}{d_{P_n\times C_m}(u)+d_{P_n\times C_m}(v)}$$
$$= 3.|E_1| + \frac{24}{7}|E_2| + 4.|E_3|$$
$$= 3.2m + \frac{24}{7}.2m + 4.(2n-1)m$$
$$= 6m + \frac{48\,m}{7} + 4\text{m}(2n-1)$$

4. Some upper bounds for the harmonic mean indices related to graph operations

In this section we discuss the topological index for the graph operation composition (lexicographic product) and Cartesian product of two connected graphs.

Definition. 4.1 [8],[16] The eccentricity $e_G(v)$ of a vertex v in a connected graph G is the greatest geodesic distance between v and any other vertex. The diameter $D(G)$ of G is defined as $d(G) = \max\{e_G(v)|\, \boldsymbol{v} \in V(G)\}$. Also the radius $rad(G)$ is defined as the $d(G) = \min\{e_G(v)|\, \boldsymbol{v} \in V(G)\}$.

For the Cartesian product of graphs, we have the results

(a) $|E(G_1 \times G_2)| = |E(G_1||V(G_2)| + |E(G_2||V(G_1)|$

(b) $d_{G_1 \times G_2}(u,v) = d_{G_1}(u) + d_{G_2}(v)$

(c) $|V(G_1 \times G_2)| = |V(G_1||V(G_2)|$

Theorem 4.2 [11] Let G_1 and G_2 be two graphs with order $n_1 and\ n_2$ and size m_1, m_2 respectively. Then $H_{MI}(G_1 \times G_2) \leq \frac{(\Delta_1+\Delta_2)^2(m_1n_2+m_2n_1)}{\delta_1+\delta_2}$ where Δ_1, Δ_2 are maximum degrees of G_1, G_2 and δ_1, δ_2 are their respective minimum degrees.

Proof. Suppose $V(G_1) = \{u_1, u_2, \dots, u_{n_1}\}$, $V(G_2) = \{v_1, v_2, \dots, v_{n_2}\}$ be the set of vertices of G_1 and G_2 respectively .

For the Cartesian product $G_1 \times G_2$,

$$|E(G_1 \times G_2)| = |E(G_1||V(G_2)| + |E(G_2||V(G_1)| = m_1n_2 + m_2n_1$$

$$H_{MI}(G_1 \times G_2) = \sum_{uv\in E(G_1\times G_2)} \frac{2d_{G_1\times G_2}(u).d_{G_1\times G_2}(v)}{d_{G_1\times G_2}(u)+d_{G_1\times G_2}(v)}$$

$$= \sum_{\substack{(u_i,v_j),(u_k,v_l)\in E(G_1\times G_2)\\ (u_i,v_j)\neq(u_k,v_l)}} \frac{2d_{G_1\times G_2}(u_i,v_j).d_{G_1\times G_2}(u_k,v_l)}{d_{G_1\times G_2}(u_i,v_j)+d_{G_1\times G_2}(u_k,v_l)}$$

$$= \sum_{\substack{(u_i,v_j),(u_k,v_l)\in E(G_1\times G_2)\\ j\neq l}} \frac{2d_{G_1\times G_2}(u_i,v_j).d_{G_1\times G_2}(u_k,v_l)}{d_{G_1\times G_2}(u_i,v_j)+d_{G_1\times G_2}(u_k,v_l)}$$

$$+ \sum_{\substack{(u_i,v_j),(u_k,v_l)\in E(G_1\times G_2)\\ i\neq l}} \frac{2d_{G_1\times G_2}(u_i,v_j).d_{G_1\times G_2}(u_k,v_l)}{d_{G_1\times G_2}(u_i,v_j)+d_{G_1\times G_2}(u_k,v_l)}$$

$$= A_1 + A_2$$

$$A_1 = \sum_{\substack{(u_i,v_j),(u_k,v_l)\in E(G_1\times G_2)\\ j\neq l}} \frac{2[d_{G_1}(u_i) + d_{G_2}(v_j)].[d_{G_1}(u_k) + d_{G_2}(v_l)]}{d_{G_1}(u_i) + d_{G_2}(v_j) + d_{G_1}(u_k) + d_{G_2}(v_l)}$$

Since,

$\Delta_i \geq d_{G_i}(u_i) for\ \ i = 1, \dots, n$ and $\delta_i \leq d_{G_i}(u_i)\ for\ \ i = 1, \dots, n$.

i.e, $A_1 \leq \sum_{u_i\in V(G_1)} \sum_{v_jv_l\in E(G_2)} \frac{2(\Delta_1+\Delta_2)(\Delta_1+\Delta_2)}{2(\delta_1+\delta_2)}$

$$= \frac{(\Delta_1+\Delta_2)^2}{\delta_1+\delta_2} m_2n_1$$

Similarly $A_2 \leq \frac{(\Delta_1+\Delta_2)^2}{\delta_1+\delta_2} m_1n_2$

$$H_{MI}(G_1 \times G_2) = A_1 + A_2 \leq \frac{(\Delta_1+\Delta_2)^2(m_1n_2+m_2n_1)}{\delta_1+\delta_2}$$

Remark. Equality holds in the case of product of cycles since , $H_{MI}(C_m \times C_n) = 8mn$ and

$$\frac{(\Delta_1+\Delta_2)^2(m_1n_2+m_2n_1)}{\delta_1+\delta_2} = \frac{(2+2)^2(mn+nm)}{2+2} = 8mn$$

5. Conclusion

In this paper, we defined the harmonic mean topological indices along with some literature in the introductory section, and obtain the values for some standard graphs and their line graphs and results connecting them in the second section. In the third section certain topological indices of some family of Cartesian products including nanotube $TUC_4(m,n)$ are evaluated. The last section deals with some upper and lower bound values for the harmonic mean indices.

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