International Journal of Research in Engineering, Science and Management
Volume-3, Issue-1, January-2020
www.ijresm.com | ISSN (Online): 2581-5792

The Applicability of Concepts of Fuzzy Set Theory

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***Abstract*: Zadeh [Zadeh, 1965] introduced the concepts of fuzzy sets in 1965, and within the next decade, Kramosil and Michalek [Kramosil & Michalek,1975] introduced the concept of fuzzy mathematical space with the assistance of continuous norms in 1975 which opened an avenue for further development of study in such spaces which have vital applications in Quan. physics particularly together with both string and epsilon ($\in^{\infty}$) theory which got and studied by EI Naschie [El Naschie, 1998]. George and Veeramani [George &Veeramani,1994,1997] modified the concept of fuzzy mathematical space introduced by Kramosil and Michalek also with the assistance of continuous norms. Recently, Chouhan and Badshah (2010) established a fixed point theorem in fuzzy metric spaces for W C Maps. The result obtained within the fuzzy mathematical space by using the notion of noncompatible maps or the property (E.A.) is very interesting.**

In this paper, I'll define in several ways the fuzzy mathematical space by given definitions about the fuzzy families, the fuzzy number, the symbolic logic, the fuzzy space, and other concepts supported that each real 'r' is replaced by fuzzy number (either triangular fuzzy number or singleton fuzzy set).

***Keywords*: Fuzzy set, Fuzzy numbers, Fuzzy metric space, C fixed point, E. A. property**

1. Introduction

A. *Important concepts about fuzzy set*

Definition 1.01. [Zadeh, 1965]

It X is a collection of objects denoted generically by x then a fuzzy set $\bar{A}$ in X is a set of order pairs:

$$\bar{A} = \left\{ \left(y, \bar{A}(y) \right) : y \in Y \right\}$$

$\bar{A}(x)$ is called the membership function or grade of membership x in $\bar{A}$ that maps X to the unit interval $[0,1]$.

Definition 1.02. [Zadeh, 1965]

The standard intersection of fuzzy sets $\bar{A}$ and $\bar{B}$ is defined as,

$$\left(\bar{B} \cap \bar{A} \right)(x) = \min \left\{ \bar{B}(x), \bar{A}(x) \right\}$$

$$= \bar{B}(x) \wedge \bar{A}(x)$$

For all $x \in X$.

Definition 1.03. [Zadeh, 1965]

The standard union of fuzzy sets $\bar{A}$ and $\bar{B}$ is defined as,

$$\left(\bar{A} \cup \bar{B} \right)(x) = \max \left\{ \bar{A}(x), \bar{B}(x) \right\}$$

$$= \bar{A}(x) \vee \bar{B}(x)$$

for all $x \in X$.

Definition 1.04. [Zadeh, 1965]

The standard complement of a fuzzy set $\bar{A}$ is defined as

$$\left(\neg \bar{A} \right)(y) = 1 - \bar{A}(y) .$$

Definition 1.05. [Zadeh, 1965]

Let $\bar{A}$ be a fuzzy set of X , the support of $\bar{A}$, denoted $S\left(\bar{A}\right)$ is the crisp set of X whose elements all have non zero membership grades in $\bar{A}$, that is,

$$S\left(\bar{A} \right) = \left\{ x \in X : \bar{A}(x) > 0 \right\}.$$

Definition 1.06. [Zadeh, 1965]

(α -cut) An α - level set of a fuzzy set $\bar{A}$ of X is a non-fuzzy (crisp) set denoted by $\bar{A}[\alpha]$, such that,

$$\overline{B}[\alpha]=\begin{cases}\left\{x\in X:\overline{B}(x)\geq\alpha\right\}, & if\,\alpha>0\\ cl.\left(S\left(\overline{B}\right)\right) & , if\,\alpha=0\end{cases}$$

Where $cl\left(S\left(\overline{A}\right)\right)$ denotes closure of the support of $\overline{A}$.

Result 1.07. [Chandra & Bector, 2005]

Let $\overline{A}$ be a fuzzy set in X with the membership function $\overline{A}(y)$. Let $\overline{A}[\beta]$ be the α -cuts off $\overline{A}$ and $\chi_{\overline{A}[\beta]}(y)$ be the characteristic function of the crisp set $\overline{A}[\beta]$ for all $\alpha\in[0,1]$

Then,

$$\overline{A}(y)=\sup_{\beta\in[0,1]}\left(\beta\wedge\chi_{\overline{A}[\beta]}(y)\right),\; y\in X$$

Given a fuzzy set $\overline{A}$ in X, one consider a special fuzzy set denoted $\alpha\overline{A}[\alpha]$ for $\alpha\in[0,1]$ whose membership function is defined as,

$$\overline{A}_{\alpha\overline{A}[\alpha]}(y)=\left(\beta\wedge\chi_{\overline{A}[\beta]}(y)\right),\; y\in X$$

and the set

$$\Lambda_{\overline{A}}=\left\{\alpha:\overline{A}(x)=\alpha,\; x\in X\right\}$$

is called the level set of $\overline{A}$. Then the above theorem states that the fuzzy set $\overline{A}$ can be expressed in the form

$$\overline{A}=\bigcup_{\alpha\in\Lambda_{\overline{A}}}\left(\alpha\,\overline{A}[\alpha]\right)$$

Where $\bigcup$ denotes the standard fuzzy union. This result is called the resolution principle of fuzzy sets. The essence of the resolution principle is that a fuzzy set $\overline{A}$ can be decomposed into fuzzy sets $\alpha\overline{A}[\alpha]$, $\alpha\in[0,1]$. Looking from a different angle, it tells that a fuzzy set $\overline{A}$ in X can be retrieved as a union of its $\alpha\overline{A}[\alpha]$ sets $\alpha\in[0,1]$. This is called the representation theorem of fuzzy sets. Thus the resolution principle and the representation theorem are the two sides of the same coin as both of them essentially tell a fuzzy set $\overline{A}$ in X can always be expressed in terms of its α -cuts without explicitly resorting to its membership function $\overline{A}(x)$.

Definition 1.08. [Chandra &Bector, 2005]

A fuzzy set $\overline{A}$ of a classical set X is called normal if there exists an $x\in X$ such that $\overline{A}(x)=1$. Otherwise $\overline{A}$ is subnormal.

Definition 1.09. [Zadeh, 1965]

A fuzzy set $\overline{A}$ of X is called convex, it $\overline{A}[\alpha]$ is a convex subset of X, for all $\alpha\in[0,1]$. That is, for any x, $y\in\overline{A}[\alpha]$, and for any $\lambda\in[0,1]$ then $\lambda x+(1-\lambda)y\in\overline{A}[\alpha]$.

Definition 1.10. [Bushera, 2006]

A fuzzy set $\overline{A}$ that $S\left(\overline{A}\right)$ contains a single point $x\in X$, with $\overline{A}(x)=1$, is referred to as a singleton fuzzy set.

Definition 1.11. [Bushera, 2006]

The empty fuzzy set of X is defined as

$$\Phi=\{(x,0):\forall x\in X\}$$

Definition 1.12. [Bushera, 2006]

The largest fuzzy set in X is defined as

$$I_X=\{(x,1):\forall x\in X\}$$

Definition 1.13. [Bushera, 2006]

The concept of continuity is the same as in other functions, that say, a function f is continuous at some number c if

$$\lim_{x\to c}f(x)=f(c)$$

for all x in range of f, that require existing $f(c)$ and $\lim_{x\to c}f(x)$. In fuzzy set theory, the condition will be

$$\lim_{x\to c}\overline{A}(x)=\overline{A}(c)$$

With x and $c\in\overline{A}$.

Definition 1.14. [Zadeh, 1965]

A fuzzy set $\overline{A}$ is said to be a bounded fuzzy set, if it α -cuts $\overline{A}[\alpha]$ are (crisp) bounded sets, for all $\alpha \in [0,1]$.

Definition 1.15. [Zadeh, 1965]

A fuzzy number $\overline{A}$ is a fuzzy set of the real line with a normal,(fuzzy) convex, and continuous membership function of bounded support.

e.g. 1.16 [Zadeh, 1965]

The following fuzzy set is fuzzy number approximately

$$"5" = \{(3,0.2),(4,0.6),(5,1.0),(6,0.7),(7,0.1)\}.$$

Remark 1.17

Let $\overline{A}$ be a fuzzy number, then $\overline{A}[\alpha]$ is a closed, convex, and compact subset of R , for all $\alpha \in [0, 1]$.

Remark 1.18

We shall use the notation $\overline{B}[\delta] = [a_1(\delta), a_2(\delta)]$, where $\overline{B}[\delta]$ is an α -cut off the fuzzy number $\overline{A}$, and $a_1 : [0,1] \to R$, $a_1(\alpha) = \min \overline{B}[\delta]$, is left-hand side function which monotone, increasing and continuous $a_2 : [0,1] \to R$, $a_2(\alpha) = \max \overline{B}[\delta]$ is right-hand side function which monotone decreasing and continuous.

Proposition 1.19

If $\delta \le \beta$, then $\overline{A}[\delta] \supset \overline{A}[\beta]$.

Proposition 1.20

The support of a fuzzy number is an open interval $(a_1(0), a_2(0))$.

Definition 1.21. [Zimmerman, 1995]

Let $\overline{A}$ be a fuzzy number. If $S\left(\overline{A}\right) = \{x\}$ then $\overline{A}$ is called a fuzzy point and we use the notion $\overline{A} = \overline{x}$. Let $\overline{A} = \overline{x}$ be a fuzzy point, it is easy to

See that $\overline{A}[\alpha] = [x, x] = \{x\}, \forall \alpha \in [0,1]$.

Definition 1.22. [Buckley, Eslami, 2005]

A fuzzy number $\overline{A}$ is called a triangular fuzzy number, where $\overline{A}$ is defined by three numbers $a_1 < a_2 < a_3$ if:

i) $\overline{A}(x) = 1$ At $x = a_2$, ($\overline{A}$ is normal)

ii) The graph of $y = \overline{A}(x')$ on $[a_1, a_2]$ is straight line from $(a_1, 0)$ to $(a_2, 1)$, also on $[a_2, a_3]$ the graph of $y = \overline{A}(x)$ is straight line from $(a_2, 1)$ to $(a_3, 0)$.

iii) $\overline{A}(x) = 0$ For $x' \le a_1$ or $x \ge a_3$.

We write $\overline{A} = (a_1, a_2, a_3)$ for triangular fuzzy number and its α -cut

$$\overline{A}[\alpha] = [(a_2 - a_1)\alpha + a_1, (a_2 - a_3)\alpha + a_3]$$

for all $\alpha \in [0,1]$.

2. Arithmetic operations on fuzzy numbers

We will define the arithmetic operations on fuzzy numbers based on the resolution principle (α -cuts).

Definition 2.01. [George, 1995]

Let $\overline{A}$ and $\overline{B}$ be two fuzzy numbers and

$$\overline{A}[\alpha] = [a_1(\alpha), a_2(\alpha)] \;,\; \overline{B}[\alpha] = [b_1(\alpha), b_2(\alpha)]$$

be α -cuts, $\alpha \in [0,1]$ of $\overline{A}$ and $\overline{B}$ respectively. Then the operation ($*$ denoted any of the arithmetic operations $(+), (-), (\cdot), (\div)$, $\wedge$, $\vee$) on fuzzy numbers $\overline{A}$ and $\overline{B}$ denoted by $\overline{A} * \overline{B}$ gives a fuzzy number in R , where

$$\overline{A} * \overline{B} = \bigcup_{\alpha} \alpha \left(\overline{A} * \overline{B} \right)[\alpha]$$

And

$$\left(\overline{A} * \overline{B} \right)[\alpha] = \overline{A}[\alpha] * \overline{B}[\alpha] \;,\; \alpha \in [0,1]$$

Here it may be remarked that the reason for $\overline{A}*\overline{B}$ to be a fuzzy number, and not just a general fuzzy set, is that $\overline{A}$ and $\overline{B}$ being fuzzy numbers, the sets $\overline{A}[\alpha]$,

$\overline{B}[\alpha], \left(\overline{A}*\overline{B}\right)[\alpha]$, are all closed intervals for all $\alpha \in [0,1]$.

In particular

$$\overline{A}[\alpha](+)\overline{B}[\alpha] = [a_1(\alpha)+b_1(\alpha), a_2(\alpha)+b_2(\alpha)]$$
$$\overline{A}[\alpha](-)\overline{B}[\alpha] = [a_1(\alpha)-b_2(\alpha), a_2(\alpha)-b_1(\alpha)]$$

Furthermore, for fuzzy numbers $\overline{A}$ and $\overline{B}$ in $\boldsymbol{R}_+$

3. The fuzzy expansion families

Definition 3.1. [2008]

The set of natural numbers is $N = \{1,2,3,...\}$.

The set of integer numbers is

$$Z = \{...,-3,-2,-1,0,1,2,3,...\}.$$

The set of rational numbers is

$$Q = \left\{ a/b : a,b \in Z, b \neq 0 \right\}.$$

In other expressions, *every terminating or recurring decimal is a rational number*.

That is, every non terminating and non-recurring decimal is an irrational number. The set of all irrational numbers is denoted by Q'. The set $R = Q \cup Q'$ is called the set of real numbers.

Remember that: If we substitute every real number $r \in R$ by a fuzzy number $\overline{r}$, such that:

If $r \in Z$, we replace r by a singleton fuzzy set $\overline{r}$. The set of all fuzzy numbers $\overline{r}, r \in Z$, will be called the family of fuzzy integer numbers and denoted by $\overline{\overline{Z}}$, where $\overline{\overline{Z}} = \{...,-\overline{2},-\overline{1},\overline{0},\overline{1},\overline{2},...\}$. The family of all fuzzy natural numbers will be $\overline{\overline{N}} = \{\overline{1},\overline{2},\overline{3},...\}$.

Because dense of rational and irrational numbers, we replace every rational r and irrational numbers r' by a triangular fuzzy number $\overline{r} = (r_1, r_2, r_3)$, it α-cut's $\overline{r}[\alpha] = [r_1(\alpha), r_2(\alpha)], \alpha \in [0,1]$, and $\overline{r}' = (r_1', r_2', r_3')$, it α-cut's $\overline{r}'[\alpha] = [r_1'(\alpha), r_2'(\alpha)]$, $\alpha \in [0,1]$ respectively.

The set of all fuzzy numbers $\overline{r}, r \in Q$ and the set of all fuzzy numbers $\overline{r}', r' \in Q'$ will be called the family of fuzzy rational numbers and the family of fuzzy irrational numbers which denoted by $\overline{\overline{Q}}$ and $\overline{\overline{Q}}'$ respectively.

The set $\overline{\overline{R}} = \overline{\overline{Q}}' \cup \overline{\overline{Q}} \cup \overline{\overline{Z}}$ will be called the family of fuzzy real numbers.

Remark 3.2

The fuzzy numbers mean here either triangular fuzzy number or singleton fuzzy set.

Definition 3.3

From definition1, we can define the following,

$i)$ $\overline{\overline{Z_0}}$ the family of all non- fuzzy zero fuzzy integer numbers. That is, for all $\overline{r} \in \overline{\overline{Z_0}}$, then $\overline{r} \neq \overline{0}$.

$ii)$ $\overline{\overline{Z_-}}$ the family of all negative fuzzy integer numbers. That is, for all $\overline{r} \in \overline{\overline{Z_-}}$, then $\overline{r} < \overline{0}$.

$iii)$ $\overline{\overline{Q_0}}$ the family of all non- fuzzy zero fuzzy rational numbers. That is, for all $\overline{r} \in \overline{\overline{Q_0}}$, then $\overline{r} \neq \overline{0}$.

$iii)$ $\overline{\overline{Q_+}}$ the family of all positive fuzzy rational numbers. That is, for all $\overline{r} \in \overline{\overline{Q_+}}$, then $\overline{r} > \overline{0}$.

$iv)$ $\overline{\overline{Q_-}}$ the family of all negative fuzzy rational numbers. That is, for all $\overline{r} \in \overline{\overline{Q_-}}$, then $\overline{r} < \overline{0}$.

$v)$ $\overline{\overline{R_+}}$ the family of all positive fuzzy real numbers. That is, for all $\overline{r} \in \overline{\overline{R_+}}$, then $\overline{r} > \overline{0}$.

$vi)$ $\overline{\overline{R_-}}$ the family of all negative fuzzy real numbers. That is, for all $\overline{r} \in \overline{\overline{R_-}}$, then $\overline{r} < \overline{0}$.

$vii)$ $\overline{\overline{N_{\overline{k}}}}$ the family of all fuzzy natural numbers which are less or equal to $\overline{k}$, where $\overline{k}$ is a positive fuzzy integer. Thus $\overline{\overline{N_{\overline{k}}}} = \{\overline{1},\overline{2},...,\overline{k}\}$.

Definition 3.4. Sharma, 1977]

The sets of the forms,

$$\{x\in R: a<x<b\},\{x\in R: a\le x\le b\},$$
$$\{x\in R: a\le x<b\},\{x\in R: a<x\le b\}$$

are called open interval, closed interval, right half-open interval and left half-open interval, and denoted by,

$(a,b),[a,b],(a,b],[a,b)$ respectively. The sets of the forms $\{x\in R: x>a\},\{x\in R: x<a\},\{x\in R: x\ge a\},\{x\in R: x\le a\}$ are called rays and denoted by $(a,\infty),(-\infty,a)$, $[a,\infty),(-\infty,a]$ respectively. The first and second rays are said to be *open rays*, and the end two rays are called the *closed rays*.

Remember that:

Suppose we have $\bar{\bar{R}}$ the family of fuzzy real numbers. The families of the forms

$$\left\{\bar{x}\in\bar{\bar{R}}:\bar{a}<\bar{x}<\bar{b}\right\},\left\{\bar{x}\in\bar{\bar{R}}:\bar{a}\le\bar{x}\le\bar{b}\right\},$$
$$\left\{\bar{x}\in\bar{\bar{R}}:\bar{a}\le\bar{x}<\bar{b}\right\},\left\{\bar{x}\in\bar{\bar{R}}:\bar{a}<\bar{x}\le\bar{b}\right\}$$

will be called open fuzzy interval, closed fuzzy interval, right half-open fuzzy interval and left half-open fuzzy interval, and will be denoted by $(\bar{a},\bar{b}),[\bar{a},\bar{b}],(\bar{a},\bar{b}],[\bar{a},\bar{b})$ respectively, if the sets at degree α, $\{x_\alpha\in R: a_\alpha<x_\alpha<b_\alpha\},\{x_\alpha\in R: a_\alpha\le x_\alpha\le b_\alpha\},\{x_\alpha\in R: a_\alpha\le x_\alpha<b_\alpha\}$, $\{x_\alpha\in R: a_\alpha<x_\alpha\le b_\alpha\}$ are open interval, closed interval, right half-open interval and left half-open interval respectively, for all $\alpha\in[0,1]$.

The families of the forms $\left\{\bar{x}\in\bar{\bar{R}}:\bar{x}>\bar{a}\right\}$ $\left\{\bar{x}\in\bar{\bar{R}}:\bar{x}<\bar{a}\right\},\left\{\bar{x}\in\bar{\bar{R}}:\bar{x}\ge\bar{a}\right\},\left\{\bar{x}\in\bar{\bar{R}}:\bar{x}\le\bar{a}\right\}$ will be called the fuzzy rays, if the sets at degree α, $\{x_\alpha\in R: x_\alpha>a_\alpha\}$, $\{x_\alpha\in R: x_\alpha<a_\alpha\}$, $\{x_\alpha\in R: x_\alpha\ge a_\alpha\},\{x_\alpha\in R: x_\alpha\le a_\alpha\}$ are rays, for all $\alpha\in[0,1]$. The first two families will be called open fuzzy rays and denoted by $(\bar{a},\bar{\infty}),(-\bar{\infty},\bar{a})$ respectively. The last two families are called closed fuzzy rays and will be denoted by $[\bar{a},\bar{\infty}),(-\bar{\infty},\bar{a}]$ respectively.

Definition 3.5. [Bhattacharya et al.,1989]

A set S is a collection of objects (or elements). It S is a set and x is an element of the set S, we say that x belongs to S, and we write $x\in S$. If x it doesn't belong to S, we on the uncertainty of Cantorian geometry and two-slit experiment. Write $x\notin S$.

Remark: Since the real numbers is essential to every set S, and the elements x of S is one form of real numbers. Hence, if we have the family of fuzzy real numbers $\bar{\bar{R}}$, the fuzzy number $\bar{x}$ will become one form of fuzzy real numbers and $\bar{\bar{S}}$ will be a family of fuzzy numbers $\bar{x}$.

4. The fuzzy metric and the fuzzy pseudo metric

Definition 4.1. [Thomas, 2000]

The distance between two points $P(x_1,y_1)$ and $Q(x_2,y_2)$ is

$$d(P(x_1,y_1),Q(x_2,y_2))=\sqrt{(\Delta x)^2+(\Delta y)^2}=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

Definition 6.2. [Thomas, 2000]

The absolute value of a number x, denoted by $|x|$ is defined by the formula,

$$|x|=\begin{cases}x, & x\ge 0\\ -x, & x<0\end{cases}$$

Note that: If we have the family of fuzzy real numbers $\bar{\bar{R}}$, and $\bar{x}\in\bar{\bar{R}}$. Let $x_\alpha\in\bar{x}[\alpha]$ off $\bar{x}$, then the absolute value of x_α a degree α will be defined by,

$$|x_\alpha|_\alpha=\begin{cases}x_\alpha, & x_\alpha\ge 0\\ -x_\alpha, & x_\alpha<0\end{cases}$$

And $|x_\alpha|_\alpha\in\overline{|\bar{x}|}[\alpha]$ α -cut of $\overline{|\bar{x}|}$.

Hence, the absolute fuzzy value of $\bar{x}$ is

$$\overline{|\bar{x}|}=\begin{cases}\bar{x}, & \bar{x}\ge 0\\ -\bar{x}, & \bar{x}<0\end{cases}$$

Definition 4.3 [Royden, 1966]

Let X be any set, a function $d: Y\times Y\to R$ is said to be a metric on X if:

1) $d(x,y)\ge 0$, for all $x,y\in X$.
2) $d(x,y)=0$ iff $x=y$.
3) $d(x,y)=d(y,x)$, for all $x,y\in X$.

4) $d(x,y) \le d(x,z) + d(z,y)$, for all $x, y, z \in X$.

A set X with a metric d is said to be a metric space and maybe denoted (X, d).

Remember that: If we have a family of fuzzy numbers $\overline{\overline{X}}$. A fuzzy function $\overline{\overline{d}} : \overline{\overline{X}} \times \overline{\overline{X}} \to \overline{\overline{R}}$ will be called a fuzzy metric on $\overline{\overline{X}}$ if satisfy:

1) $\overline{d}(\overline{x}, \overline{y}) \ge \overline{0}$, for all $\overline{x}, \overline{y} \in \overline{\overline{X}}$.
2) $\overline{d}(\overline{x}, \overline{y}) = \overline{0}$ iff $\overline{x} = \overline{y}$.
3) $\overline{d}(\overline{x}, \overline{y}) = \overline{d}(\overline{y}, \overline{x})$, for all $\overline{x}, \overline{y} \in \overline{\overline{X}}$.
4) $\overline{d}(\overline{x}, \overline{y}) \le \overline{d}(\overline{x}, \overline{z}) + \overline{d}(z, y)$, for all $\overline{x}, \overline{y}, \overline{z} \in \overline{\overline{X}}$.

A family of fuzzy numbers $\overline{\overline{X}}$ with a fuzzy metric $\overline{\overline{d}}$ will be called a fuzzy metric space and will be denoted $\left(\overline{\overline{X}}, \overline{\overline{d}}\right)$.

5. Conclusion

In this paper I am willing to summarize the fuzzy mathematical space about the fuzzy families, the fuzzy number, the symbolic logic, the fuzzy space, and other concepts supported that each real expression fuzzy number related to respective topic. This paper will help the researchers to look the field of Applicability of Fuzzy Set Theory at a glance.

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