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Laplace Transform Deconvolution and its Application to the Solution of the Hyperbolic Diffusivity Equation Under Wellbore Boundary Conditions

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Abstract*: The deconvolution of variable rate and variable pressure data in well test analysis into, variable rate at constant pressure or variable pressure at constant rate is quite easily achieved in the Laplace domain operations. Issues involved with other spline-based methods are that: the piecewise linear function is discontinuous over its first derivative and as such will require many knots for better approximation of nonlinear trends, and the cubic spline has a higher tendency to oscillate around discontinuities. The major problems associated with the existing solution models of the parabolic diffusivity equation is the negligence of fluid density and possible inertia effects, and of course the inherent assumption of infinite speed of pressure propagation in the base transport equation, which implies inaccurate early times description. The first part of this work involves the analysis of variable rate and variable pressure data by the Laplace domain deconvolution process using the second order spline. The second order spline, when used with other functions can accurately transform sampled data into Laplace domain, other approaches found in the literature are used with the spline methods to handle discontinuities and noise in data, which make the method an algorithm that accurately transforms sampled data into Laplace domain. Secondly, is the extension of the convenience of the Laplace domain operations to solving the dimensionless radial flow hyperbolic diffusivity equation for infinite-acting systems. The hyperbolic diffusivity equation is a telegrapher's model representing the reservoir fluid dynamics. Fluid inertia was taken into consideration in the base transport equation leading to the development of the hyperbolic diffusivity equation, thus, there is no assumption of infinite propagation speed of pressure disturbance. The Heaviside expansion method was further used to approximate the transformed variables before they are inverted back to time domain. Results were presented in figures to validate the proposed solution to the hyperbolic diffusivity equation and to compare with the solution offered to the parabolic diffusivity equation for various values of dimensionless (dummy) variable. The proposed solution to the hyperbolic flow equation captures the early time behaviour better, hence, better represents the reservoir system.

***Keywords*: Boundary Value Problem (IBVP), Initial conditions, Inner boundary conditions, Parabolic and Elliptic diffusivity equations, Volterra Integral Equation of the first kind.**

1. Introduction

The deconvolution of variable rate and variable pressure data in Petroleum engineering, into variable pressure at constant rate or variable rate at constant pressure based on the Duhamel's principle (the convolution integral for pressure and rate functions) requires the transformation of time dependent variables into Laplace domain, basically accomplished using the piecewise linear interpolation algorithm by fitting straight lines through successive knots, which has been found to be approximate for nonlinear trends in sampled data. The Laplace domain deconvolution method can be greatly extended in application if real time functions, which are only known as a table of the functions f(t) versus time (t) values, can be converted to Laplace space forms. The process of numerical Laplace transformation is relatively straight forward when the function f(t) is well behaved (Roumboutsos and Stewart, 1988). The piecewise linear approximation was introduced by Roumboutsos and Stewart in their work on direct deconvolution and convolution algorithm for well test analysis, to take Laplace transform of the data functions directly, on careful selection of the knots. Issues of oscillations due to noise in measured data are usually encountered and instabilities in numerical inversion processes because of discontinuities usually accompany the spline-based deconvolution methods. It is therefore required to have an adequate algorithm to transform the piecewise-continuous sampled data into the Laplace space and an appropriate numerical Laplace inversion algorithm capable of processing the exponential contributions caused by the tabulated data to exploit the potential of Laplace domain operations (Mahmood, 2012). To smoothen the deconvolved pressure response, an adaptive approach using a Gaussian and Epanechnikov kernel regression had been proposed. Also, the boundary mirroring approach was introduced by Mahmood to eliminate the effect of instabilities caused by discontinuities.

Most importantly, in this work, is the application of the famous Laplace integral transformation method to solving the initial and boundary value problem (IBVP) presented by the hyperbolic diffusivity equation. The amount of literature regarding the use of the parabolic diffusivity equation in modelling fluid flow through porous media is surplus (Blasingame, 1997), but also, a good number of literatures such as that of Baumeister (1969) are of the idea that the resulting model governing the flow of fluid through porous media is the hyperbolic diffusivity equation. The transport equation that was used to derive the parabolic diffusivity equation is the Darcy's law, which omits the possibility of the effect of inertia of the fluid. The problem (hyperbolic diffusivity equation) treated here was formulated based on the usual material balance (continuity) equation and the equation of state for slightly compressible liquid underlying the development of the parabolic diffusivity equation, but the transport equation was modified by including the effect of the fluid inertia and density, which was omitted in Darcy's law. It is observed therefore, that the derivation of the parabolic diffusivity equation assumes that the pressure propagates at an infinite speed, and as such, the resulting solution models cannot be perfectly accurate to describe the early times flow behaviour of fluids through porous media. Due to the modification of the transport equation, the resulting transport equation utilised in the derivation of the hyperbolic diffusivity equation is Newton-like and Darcy-like in form; it is Newton-like because it is related to the product of mass and acceleration of the fluid and Darcy-like because if the density is neglected it becomes the well-known Darcy law (Jeffrey, 2014). The hyperbolic diffusivity equation is also referred to as the 'generalised diffusivity equation' because, when the speed of propagation is infinite, the resulting equation is the parabolic diffusivity equation, and for steady state flow, the equation becomes elliptic.

Van Everdingen and Hurst (1949) introduced the application of convolution/deconvolution in the Laplace domain using the Duhamel's principle to obtain a solution for dimensionless wellbore pressure drop. They specifically highlighted the conveniences since the convolution of two functions results in an algebraic product of the functions in the Laplace transform domain. This Laplace domain operation provides a convenient means of generating analytical solutions for many variable-rate and variable pressure problems encountered in the field of petroleum engineering. This work therefore introduces the use of second-order spline interpolation algorithm in transforming sampled data into Laplace domain in well test convolution and deconvolution analyses and, also extends existing mathematical procedures to obtain the dimensionless solutions to the generalised (hyperbolic) diffusivity equation in the Laplace domain, considering an infinite acting reservoir which only allows us to measure the reservoir properties at the inner (wellbore) boundary. Results are compared with those obtained by Van Everdingen and Hurst (1949), showing clear distinction at early flow periods.

2. Mathematical basis of deconvolution

The Laplace transform $F(s)$ of a real function f(t):

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt, \ \ \forall\, t > 0 \qquad (1)$$

The Laplace transform of two convoluted functions *f(t)* and *g(t)* yields the product of the transforms of the two functions.

$$\mathcal{L}\{f(t) * g(t)\} = \mathcal{L}\left\{\int_0^t f(\tau)g(t-\tau)d\tau\right\} = \boldsymbol{F}(s)\boldsymbol{G}(s) \qquad (2)$$

The Duhamel's principle (Duhamel, 1833), used for solving pressure drop function, $\Delta p\ (r, t)$, at position r and time t due to variable rate (q_D), in the analysis of variable rate data in flow through porous media, is simply the application of eq. (2).

$$\Delta P(r,t) = \int_0^t q_D(\tau)\Delta P'_u(r, t-\tau)d\tau \qquad (3)$$

The process of recovering the impulse or unit rate function $\Delta P_u(r,\ t)$ from measured variable rate response, $\Delta P(r,\ t)$, and variable rate, $q_D(t)$, is the reversal of the effect of convolution known as deconvolution. This implies that we are trying to solve a *Volterra Integral Equation* of the first kind.

By taking the Laplace transform of (3):

$$\Delta\bar{P}(s) = \bar{q}_D(s)\Delta\bar{P}'_u(s) \qquad (4)$$

But:

$$\mathcal{L}\{P'(t)\} = s\bar{P}(s) - P(0) \qquad (5)$$

Therefore,

$$\mathcal{L}\{\Delta P'_u(t)\} = \Delta\bar{P}'_u(s) = s\Delta\bar{P}_u(s) - \Delta P_u(0) \qquad (6)$$

But at time, t=0, $\Delta P_u = 0$, therefore,

$$\Delta\bar{P}'_u(s) = s\Delta\bar{P}_u(s) \qquad (7)$$

Then, (4) becomes:

$$\Delta\bar{P}(s) = \bar{q}_D(s).\, s\Delta\bar{P}_u(s) \qquad (8)$$

$$\Delta\bar{P}_u(s) = \frac{\Delta\bar{P}(s)}{s\bar{q}_D(s)} \qquad (9)$$

On inversion back to time domain:

$$\Delta P_u(t) = \mathcal{L}^{-1}\left\{\frac{\Delta\bar{P}(s)}{s\bar{q}_D(s)}\right\} \qquad (10)$$

This is the deconvolution process in which the constant rate response $\Delta P_u(t)$ has been recovered from the measured rate and pressure data. Once recovered, this constant rate response can then be analysed using the standard drawdown technique. The application of (10) requires that measured field data, $\Delta P(r, t)$ and $q(t)$ be transformed to the Laplace transform domain using a proper approximating function and in addition, a suitable numerical Laplace transform inversion algorithm which effectively handles discontinuities and noise, to carry out the inversion process.

3. Second order spline based deconvolution

If $S(t)$ is given over the range $a \leq t \leq b$ with knots defined by:

$$a = t_0 < t_1 < t_2 \dots < t_n \tag{11}$$

The second order spline in each subinterval can be written as follows:

$$S_i(t) = at_i^2 + bt_i + c, \qquad i = 1,2,\dots,n \tag{12}$$

Where, S represents measured dependent variable data such as rate or pressure as a function of time, t. The coefficient a, b *and* c are constants and different for each subinterval.

Second order spline function for each subinterval $t_{i-1} \leq t \leq t_i$ with values y_{i-1} and y_i at t_{i-1} and t_i respectively for pressure:

$$P(t) = \frac{\ddot{m}}{2}(t_{i-1} - t)(t_i - t) + \frac{P_i(t - t_{i-1}) + P_{i-1}(t_i - t)}{(t_i - t_{i-1})} \tag{13}$$

Where, $\ddot{m}$ is the second derivative of the function. (13) can be rearranged as follows:

$$P(t) = \frac{\ddot{m}t^2}{2} + \left[\frac{P_i - P_{i-1}}{t_i - t_{i-1}} - \frac{\ddot{m}(t_i + t_{i-1})}{2}\right]t + \left(\frac{\ddot{m}t_i t_{i-1}}{2} + \frac{P_{i-1}t_i - P_i t_{i-1}}{t_i - t_{i-1}}\right) \quad i = 2,3,\dots,n \tag{14}$$

Applying (1) to (14) over the interval $t_1 < t < t_n$ for tabulated dependent variable data, such as pressure or production rate, gives:

$$\mathcal{L}\{P(t)\} = \bar{P}(s) = \int_0^\infty P_i(t)e^{-st}dt = \int_0^{t_1} P_i(t)e^{-st}dt + \int_{t_1}^{t_n} P_i(t)e^{-st}dt + \int_{t_n}^{\infty} P_i(t)e^{-st}dt = \int_{t_1}^{t_n} P_i(t)e^{-st}dt \tag{15}$$

But because the terms $\int_0^{t_1} P_i(t)e^{-st}dt$ and $\int_{t_n}^{\infty} P_i(t)e^{-st}dt$ are zero for a set of tabulated data within the interval $t_1 < t < t_n$, they will therefore vanish.

substituting (14) into (15) yields the following Laplace transforms:

For the first term:

$$\mathcal{L}\left\{\frac{\ddot{m}t^2}{2}\right\} = \int_{t_1}^{t_n} \frac{\ddot{m}t^2}{2}e^{-st}dt = \int_{t_1}^{t_2} \frac{\ddot{m}t^2}{2}e^{-st}dt + \int_{t_2}^{t_3} \frac{\ddot{m}t^2}{2}e^{-st}dt + \cdots + \int_{t_{n-1}}^{t_n} \frac{\ddot{m}t^2}{2}e^{-st}dt \tag{16}$$

$$\mathcal{L}\left\{\frac{\ddot{m}t^2}{2}\right\} = \sum_{i=2}^{n} \frac{\ddot{m}}{2}\left\{\left[-\frac{t_i^2}{s} - \frac{2t_i}{s^2} - \frac{2}{s^3}\right]e^{-st_i} - \left[-\frac{t_{i-1}^2}{s} - \frac{2t_{i-1}}{s^2} - \frac{2}{s^3}\right]e^{-st_{i-1}}\right\} \tag{17}$$

For the second term:

$$\mathcal{L}\left\{\left[\frac{P_i - P_{i-1}}{t_i - t_{i-1}} - \frac{\ddot{m}(t_i + t_{i-1})}{2}\right]t\right\} = \int_{t_1}^{t_n} \left[\frac{P_i - P_{i-1}}{t_i - t_{i-1}} - \frac{\ddot{m}(t_i + t_{i-1})}{2}\right]te^{-st}dt = \int_{t_1}^{t_2} \left[\frac{P_i - P_{i-1}}{t_i - t_{i-1}} - \frac{\ddot{m}(t_i + t_{i-1})}{2}\right]te^{-st}dt + \int_{t_2}^{t_3} \left[\frac{P_i - P_{i-1}}{t_i - t_{i-1}} - \frac{\ddot{m}(t_i + t_{i-1})}{2}\right]te^{-st}dt + \cdots + \int_{t_{n-1}}^{t_n} \left[\frac{P_i - P_{i-1}}{t_i - t_{i-1}} - \frac{\ddot{m}(t_i + t_{i-1})}{2}\right]te^{-st}dt \tag{18}$$

$$\mathcal{L}\left\{\left[\frac{P_i - P_{i-1}}{t_i - t_{i-1}} - \frac{\ddot{m}(t_i + t_{i-1})}{2}\right]t\right\} = \sum_{i=2}^{n} \left[\frac{P_i - P_{i-1}}{t_i - t_{i-1}} - \frac{\ddot{m}(t_i + t_{i-1})}{2}\right]\left\{\left(-\frac{t_i}{s} - \frac{1}{s^2}\right)e^{-st_i} - \left(-\frac{t_{i-1}}{s} - \frac{1}{s^2}\right)e^{-st_{i-1}}\right\} \tag{19}$$

For the third term:

$$\mathcal{L}\left\{\left(\frac{\ddot{m}t_i t_{i-1}}{2}+\frac{P_{i-1}t_i-P_i t_{i-1}}{t_i-t_{i-1}}\right)\right\}$$
$$=\int_{t_1}^{t_n}\left(\frac{\ddot{m}t_i t_{i-1}}{2}+\frac{P_{i-1}t_i-P_i t_{i-1}}{t_i-t_{i-1}}\right)e^{-st}dt$$
$$=\int_{t_1}^{t_2}\left(\frac{\ddot{m}t_i t_{i-1}}{2}+\frac{P_{i-1}t_i-P_i t_{i-1}}{t_i-t_{i-1}}\right)e^{-st}dt$$
$$+\int_{t_2}^{t_3}\left(\frac{\ddot{m}t_i t_{i-1}}{2}+\frac{P_{i-1}t_i-P_i t_{i-1}}{t_i-t_{i-1}}\right)e^{-st}dt+\cdots$$
$$+\int_{t_{n-1}}^{t_n}\left(\frac{\ddot{m}t_i t_{i-1}}{2}+\frac{P_{i-1}t_i-P_i t_{i-1}}{t_i-t_{i-1}}\right)e^{-st}dt \qquad (20)$$

$$\mathcal{L}\left\{\left(\frac{\ddot{m}t_i t_{i-1}}{2}+\frac{P_{i-1}t_i-P_i t_{i-1}}{t_i-t_{i-1}}\right)\right\}=\sum_{i=2}^{n}\left(\frac{\ddot{m}t_i t_{i-1}}{2}+\frac{P_{i-1}t_i-P_i t_{i-1}}{t_i-t_{i-1}}\right)\left\{-\frac{1}{s}\left(e^{-st_i}-e^{-st_{i-1}}\right)\right\} \qquad (21)$$

For $i = 2, 3, \ldots, n$ at $t_1 < t < t_n$.

4. Problem formulation

For a physical reservoir system of infinite radius R with a centered well of radius r, the following *simplifying assumptions* are made:

1. The reservoir is homogeneous and isotropic with respect to permeability.
2. The formation is completely saturated with a single incompressible fluid.
3. Constant and pressure independent rock and fluid properties.
4. The well is completed across the entire formation thickness to assume a fully radial flow around the wellbore.
5. Negligible gravity forces.

A. *Mathematical model*

1) *Governing Equations*

Continuity equation:

$$\nabla.(\rho\vec{v})=\frac{\partial(\rho v_x)}{\partial x}+\frac{\partial(\rho v_y)}{\partial y}+\frac{\partial(\rho v_z)}{\partial z}=-\frac{\partial(\phi\rho)}{\partial t} \qquad (22)$$

Transport Equation:

By consolidating the idea of the generalised Darcy and Navier-Stokes equations, we arrived at a hydrodynamic equivalent to Newton's second law of motion with the aim of creating a new set of transport equations that account for the effect of fluid density (inertia) and, also results to the generalized Darcy's equation when the inertia tends to zero. (Oroveanu *et.al.*, 1959; Pascal, 1986):

$$\frac{\rho}{\phi}\frac{d\vec{v}}{dt}=-\nabla P-\frac{\mu}{k}\vec{v} \qquad (23)$$

Equation of state for slightly compressible fluid:

$$\rho=\rho_o e^{(P-P_o)} \qquad (24)$$

Formation compressibility:

$$c_f=\frac{1}{\phi}\frac{\partial\phi}{\partial P} \qquad (25)$$

Fluid compressibility:

$$c=\frac{1}{\rho}\frac{\partial\rho}{\partial P} \qquad (26)$$

Total compressibility:

$$c_t=c+c_f \qquad (27)$$

Resulting Models:

The flow models (hyperbolic diffusivity equations) result from the combination of equations (23) to (27):

Linear:

$$\frac{\partial^2 P}{\partial x^2}+\frac{\partial^2 P}{\partial y^2}+\frac{\partial^2 P}{\partial z^2}=\rho c_t\frac{\partial^2 P}{\partial t^2}+\frac{\mu\phi c_t}{k}\frac{\partial P}{\partial t} \qquad (28)$$

Radial:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial P}{\partial r}\right)=\frac{1}{\propto^2}\frac{\partial^2 P}{\partial t^2}+\frac{1}{\eta}\frac{\partial P}{\partial t} \qquad (29)$$

Where,

$$\propto^2=\frac{1}{\rho c_t} \qquad (30)$$

$$\eta=\frac{k}{\mu\phi c_t} \qquad (31)$$

In terms of dimensionless variables:

$$\frac{1}{r_D}\frac{\partial}{\partial r_D}\left(r_D\frac{\partial P_D}{\partial r_D}\right)=\tau^2\frac{\partial^2 P_D}{\partial t_D^2}+\frac{\partial P_D}{\partial t_D} \qquad (32)$$

Where:

$$P_D = \frac{2\pi kh(P_i - P)}{\mu q} \quad (33)$$

$$r_D = \frac{r}{r_w} \quad (34)$$

$$r_{eD} = \frac{r_e}{r_w} \quad (35)$$

$$t_D = \frac{\eta t}{r_w^2} \quad (36)$$

$$\eta = \frac{k}{\emptyset \mu c_t} \quad (37)$$

$$\tau = \frac{\eta}{\propto r_w} \quad (38)$$

5. Solution model

Equation (32) can also be expressed as:

$$\frac{\partial^2 P_D}{\partial r_D^2} + \frac{1}{r_D}\frac{\partial P_D}{\partial r_D} = \tau^2 \frac{\partial^2 P_D}{\partial t_D^2} + \frac{\partial P_D}{\partial t_D} \quad (39)$$

A. Laplace domain solution

I.Cs:

$$P_D(r_D, t_D = 0) = 0 \quad (40)$$

$$\frac{\partial P_D}{\partial t_D}(r_D, t_D = 0) = 0 \quad (41)$$

Taking the Laplace transform of (39):

$$\frac{\partial^2 \bar{P}_D}{\partial r_D^2}(r_D, s) + \frac{1}{r_D}\frac{\partial \bar{P}_D}{\partial r_D}(r_D, s) = \tau^2 \left[s^2 \bar{P}_D(r_D, s) - sP_D(r_D, t = 0) - \frac{\partial P_D}{\partial t_D}(r_D, t = 0)\right] + s\bar{P}_D(r_D, s) - P_D(r_D, 0) \quad (42)$$

Inserting the initial conditions:

$$\frac{\partial^2 \bar{P}_D}{\partial r_D^2}(r_D, s) + \frac{1}{r_D}\frac{\partial \bar{P}_D}{\partial r_D}(r_D, s) - (\tau^2 s^2 + s)\bar{P}_D(r_D, s) = 0 \quad (43)$$

Transformation to modified Bessel equation

$$z^2 \frac{\partial^2 \bar{P}_D}{\partial z^2}(r_D, s) + z\frac{\partial \bar{P}_D}{\partial z}(r_D, s) - z^2 \bar{P}_D(r_D, s) = 0 \quad (44)$$

Where:

$$z^2 = r_D^2(\tau^2 s^2 + s) \quad (45)$$

1) Solution for terminal rate inner boundary condition

$$P_D = \frac{2\pi kh(P_i - P)}{\mu q} \quad (46)$$

B.C:

$$r_D \frac{\partial P_D}{\partial r_D} = -1, \qquad when\ r_D = 1, \forall\ t_D > 0 \quad (47)$$

General solution to (44):

$$\bar{P}_D(r_D, s) = AI_0\left(r_D\sqrt{\tau^2 s^2 + s}\right) + BK_0\left(r_D\sqrt{\tau^2 s^2 + s}\right) \quad (48)$$

$A = 0$, for pressure to remain finite as $r_D \to \infty$

From the boundary condition:

$$r_D \frac{\partial \bar{P}_D}{\partial r_D} = -\frac{1}{s} = \frac{\partial \bar{P}_D}{\partial r_D}, \qquad when\ r_D = 1 \quad (49)$$

Inserting the boundary condition and solving for the constant:

$$\bar{P}_D(r_D, s) = \frac{K_0\left(r_D\sqrt{\tau^2 s^2 + s}\right)}{s\sqrt{\tau^2 s^2 + s}. K_1\left(\sqrt{\tau^2 s^2 + s}\right)} \quad (50)$$

At the wellbore, $r_D = 1$, therefore,

$$\bar{P}_D(1, s) = \frac{K_0\left(\sqrt{\tau^2 s^2 + s}\right)}{s\sqrt{\tau^2 s^2 + s}. K_1\left(\sqrt{\tau^2 s^2 + s}\right)} \quad (51)$$

For *very small* t_D:

$$K_O(z) \approx K_1(z) = \sqrt{\frac{\pi}{2z}}e^{-z} \quad (52)$$

Therefore,

$$\bar{P}_D(1, s) = \frac{1}{s\sqrt{\tau^2 s^2 + s}} \quad (53)$$

By *Heaviside* expansion and inverse transform:

$$P_D(1, t_D) \approx \frac{t_D^{1/2}}{\Gamma\left(\frac{1}{2} + 1\right)} - \frac{t_D^{-1/2}\tau^2}{2\,\Gamma\left(-\frac{1}{2} + 1\right)} + \frac{3t_D^{-3/2}\tau^4}{8\,\Gamma\left(-\frac{3}{2} + 1\right)} - \frac{5t_D^{-5/2}\tau^6}{16\,\Gamma\left(-\frac{5}{2} + 1\right)} + \frac{105t_D^{-7/2}\tau^8}{384\,\Gamma\left(-\frac{1}{2} + 1\right)} - + \cdots \quad (54)$$

$$P_D(1,t_D) \approx 2\sqrt{\frac{t_D}{\pi} - \frac{\tau^2}{2\sqrt{\pi t_D}} - \frac{3\tau^4}{16\sqrt{\pi}t_D{}^{3/2}} - \frac{15\tau^6}{64\sqrt{\pi}t_D{}^{5/2}} - \frac{1575\tau^8}{3072\sqrt{\pi}t_D{}^{7/2}} - \cdots} \quad (55)$$

For *larger* t_D:

$$K_0(z) \approx \frac{1}{2}\ln\left(\frac{4}{e^{2\gamma}}\frac{1}{z^2}\right) \quad (56)$$

$$K_1(z) \approx \frac{1}{z} \quad (57)$$

Therefore,

$$\bar{P}_D(1,s) = \frac{\frac{1}{2}\ln\left(\frac{4}{e^{2\gamma}}.\frac{1}{\tau^2 s^2 + s}\right)}{s} \quad (58)$$

By *Heaviside* expansion and inversion back to time domain:

$$P_D(1,t_D) = \ln 2 - \gamma - \frac{1}{2}(-\gamma - \ln t_D) \quad (59)$$

$$P_D(1,t_D) = \ln 2\sqrt{t_D} - \frac{1}{2}\gamma \quad (60)$$

2) *Solution for terminal pressure inner boundary condition*

$$P_D = \frac{P_i - P}{P_i - P_{wf}} \quad (61)$$

B.C:

$$P_D(r_D = 1, t_D) = 1 \qquad \forall\, t_D > 0 \quad (62)$$

Therefore,

$$\bar{P}_D(1,s) = \frac{1}{s} \quad (63)$$

Solution:

$$\bar{P}_D(r_D,s) = \frac{K_0\left(r_D\sqrt{\tau^2 s^2 + s}\right)}{sK_0\left(\sqrt{\tau^2 s^2 + s}\right)} \quad (64)$$

$$\bar{q}_D = -r_D\frac{\partial \bar{P}_D}{\partial r_D} \quad (65)$$

From (63),

$$\frac{\partial \bar{P}_D}{\partial r_D}(r_D,s) = -\frac{\sqrt{\tau^2 s^2 + s}.K_1\left(r_D\sqrt{\tau^2 s^2 + s}\right)}{sK_0\left(\sqrt{\tau^2 s^2 + s}\right)} \quad (66)$$

Therefore,

$$\bar{q}_D(r_D,s) = \frac{r_D\sqrt{\tau^2 s^2 + s}.K_1\left(r_D\sqrt{\tau^2 s^2 + s}\right)}{sK_0\left(\sqrt{\tau^2 s^2 + s}\right)} \quad (67)$$

At the inner boundary (wellbore):

$$\bar{q}_D(1,s) = \frac{\sqrt{\tau^2 s^2 + s}.K_1\left(\sqrt{\tau^2 s^2 + s}\right)}{sK_0\left(\sqrt{\tau^2 s^2 + s}\right)} \quad (68)$$

For *very small* t_D, (68) reduces to:

$$\bar{q}_D(1,s) = \frac{\sqrt{\tau^2 s^2 + s}}{s} \quad (69)$$

But,

$$Q_D(1,t_D) = \int_0^{t_D} q_D(1,t_D)dt_D \quad (70)$$

Taking the Laplace transform of (70):

$$\bar{Q}_D(1,s) = \frac{\bar{q}_D}{s}(1,s) \quad (71)$$

Therefore,

$$\bar{Q}_D(1,s) = \frac{\sqrt{\tau^2 s^2 + s}}{s^2} = \frac{\sqrt{\tau^2 s + 1}}{s^{3/2}} \quad (72)$$

By *Heaviside* expansion and inverse transform to time domain:

$$Q_{Dc}(1,t_D) \approx \frac{t_D{}^{1/2}}{\Gamma\left(\frac{1}{2}+1\right)} + \frac{t_D{}^{-1/2}\tau^2}{2\,\Gamma\left(-\frac{1}{2}+1\right)} - \frac{t_D{}^{-3/2}\tau^4}{8\,\Gamma\left(-\frac{3}{2}+1\right)} + \frac{t_D{}^{-5/2}\tau^6}{16\,\Gamma\left(-\frac{5}{2}+1\right)} - \frac{5t_D{}^{-7/2}\tau^8}{128\,\Gamma\left(-\frac{1}{2}+1\right)} - + \cdots \quad (73)$$

$$Q_{Dc}(1,t_D) \approx 2\sqrt{\frac{t_D}{\pi} + \frac{\tau^2}{2\sqrt{\pi t_D}} + \frac{\tau^4}{16\sqrt{\pi}t_D{}^{3/2}} + \frac{3\tau^6}{64\sqrt{\pi}t_D{}^{5/2}} + \frac{75\tau^8}{1024\sqrt{\pi}t_D{}^{7/2}} + \cdots} \quad (74)$$

As seen before, solution to the hyperbolic diffusivity equation for larger time approximation is expected to yield the same result as that of its parabolic counterpart. This implies that the solution model for the parabolic equation converges to that of its hyperbolic counterpart after some time from the beginning of flow.

6. Results

The dimensionless pressure solutions to the famous parabolic diffusivity equation (the case for τ=0) can be found in the work of Van A.F and Hurst W (1949), which is compared with the solutions obtained in this work. Furthermore, log-log plots of the P_D alongside P_D(Der) versus t_D for the various values of τ are shown for more illustration of the distinction.

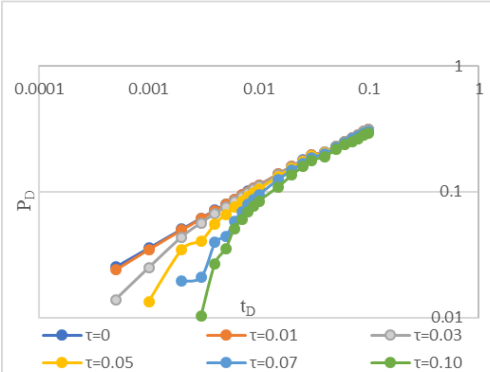

Fig. 1. Log-log plot of P_D versus t_D for various values of τ

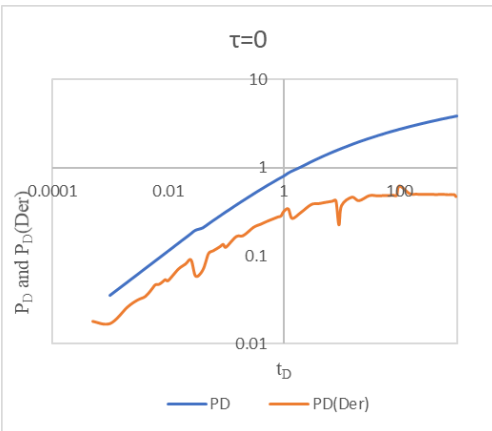

Fig. 2. Log-log plot of P_D and P_D(Der) versus t_D for τ = 0

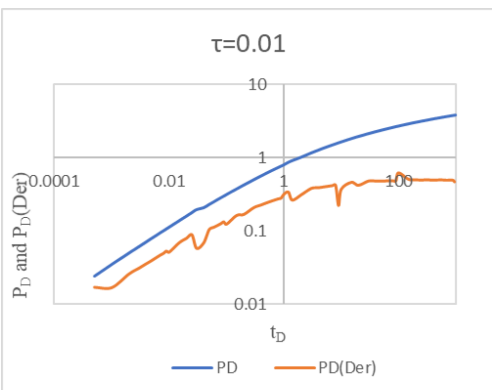

Fig. 3. Log-log plot of P_D and P_D(Der) versus t_D for τ = 0.01

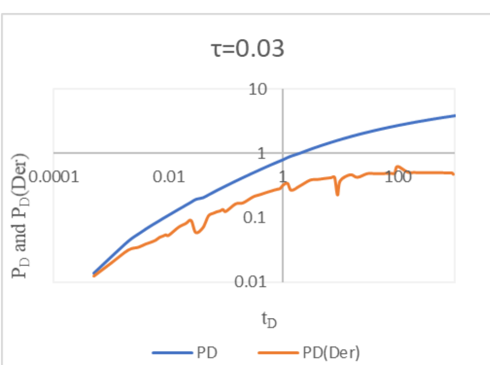

Fig. 4. Log-log plot of P_D and P_D(Der) versus t_D for τ = 0.03

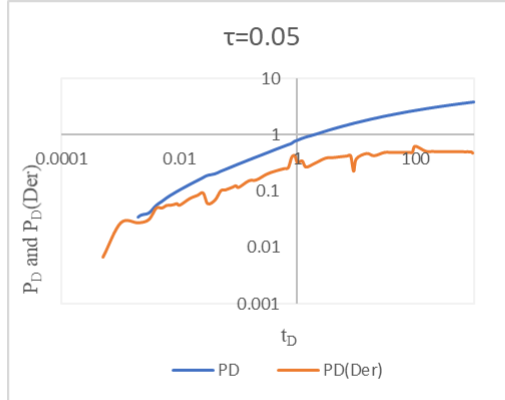

Fig. 5. Log-log plot of P_D and P_D(Der) versus t_D τ = 0.05

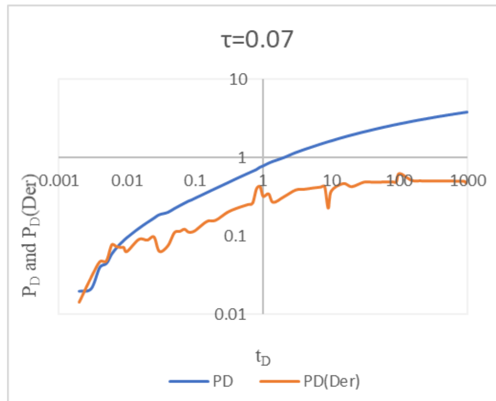

Fig. 6. Log-log plot of P_D and P_D(Der) versus t_D for τ = 0.07

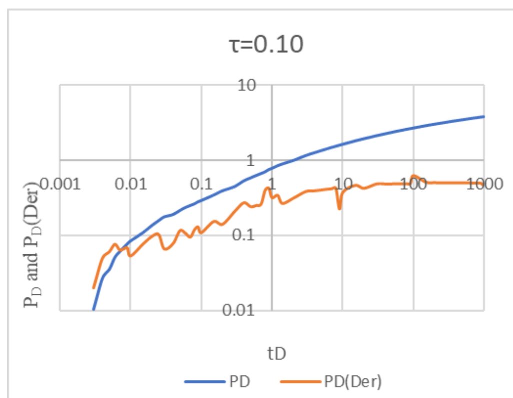

Fig. 7. Log-log plot of P_D and P_D(Der) versus t_D for τ = 0.10

7. Discussion

The solutions obtained in this work yield the same results for the larger times approximation as those of the work of Van Everdingen and Hurst (1949). These solutions are useful for pressure profile analysis which is vital for production optimization in the oil and gas industry.

Also, the derivative plots show the stabilized dimensionless pressure of 0.5 which we wish to maintain. This is the region which indicates the ideal fully radial flow for the infinite acting system that is producing clean oil.

As presented in fig. 1, it is observed that the inherent assumption of infinite speed of pressure propagation through the fluid in Darcy's Law due to the negligence of possible inertia effect will have some effects on the results obtained. As clearly demonstrated, the solutions to the radial flow parabolic diffusivity equation will only converge to that of its hyperbolic counterpart after sufficient time. It is therefore of the essence to model with the hyperbolic diffusivity equation if the description of the early time flow behaviour of the reservoir system is vital in any reservoir engineering analysis.

8. Conclusion

The extension of the convenience of the Laplace domain deconvolution to solving the hyperbolic diffusivity equation, which is an extension on the classical work of Van Everdingen and Hurst (1949), clearly shows that the model (hyperbolic diffusivity equation) can accurately model the reservoir flow behaviour and better represents the early times flow nature of the reservoir system.

Nomenclature

$\mathcal{L}$ = Laplace operator.
s = Laplace transform parameter.
$\forall$ = for all.
$\Delta P(r,t)$ = pressure drop function.
ΔP_u = impulse (unit or constant rate) function drop.
$\Delta P'_u$ = derivative of the impulse function drop.
r_w = wellbore radius.
μ = fluid viscosity.
$\emptyset$ = formation porosity.
k = formation permeability.
h = pay thickness.
$\mathrm{r_D}$ = dimensionless radius.
$\mathrm{t_D}$ = dimensionless time.
τ = dummy variable.
$\mathrm{q_D}$ = dimensionless flow rate.
Q_D = dimensionless cumulative production.
$\mathrm{P_D}$= dimensionless pressure.
$\bar{q}_D$ = transformed dimensionless rate.
$\bar{Q}_D$ = transformed dimensionless cumulative production.
$\bar{P}_D$ = transformed dimensionless pressure.
∇ = div operator.
$\vec{v}$ = directional velocity.
ρ = density.
c_f = formation compressibility.
c = fluid compressibility.
c_t = total compressility.
η = hydraulic diffusivity.
$\propto$= speed of sound through fluid.
A, B = constants in the general solution to the modified Bessel equation.
I_0 = zeroth order modified Bessel function of the first kind.
K_0 = zeroth order modified Bessel function of the second kind.
K_1 = first order modified Bessel function of the second kind.
e = exponentiation.
∞ = infinity.
$\pi \approx 3.142$
γ = Euler's constant ≈ 0.5772.
∂ = partial differential.
$\int$ = integral sign.

References

[1] Abramowitz M and Stegun A. 1972. Handbook of Mathematical Functions. Dover Publications, Mineola, New York.
[2] Baumeister K.J. and Hamill T.D. (1969). "Hyperbolic heat-conduction equation-A solution for the semi-infinite body problem". Journal of Heat Transfer, 91 No.4, 543-548.
[3] Baygun, B., Kuchuk, F.J., and Arikan, O. (1997). "Deconvolution Under Normalized Autocorrelation Constraints". SPEJ (September) 246.
[4] Blasingame T.A. (1997). "Derivation of the diffusivity equation for single-phase liquid flow (in any flow geometry)", Petroleum Engineering 620 Course Notes.
[5] Cinar M., Ilk, D., Onur, M., Valko, P.P, and Blasingame, T.A. (2006). "A Comparative Study of Recent Robust Deconvolution Algorithms for Well-Test and Production-Data Analysis". Paper SPE 102575 presented at the SPE Annual Technical Conference and Exhibition, San Antonio, Tx, 24-27 Sep.
[6] Dacunha J.J. (2014). "Solutions and Generalizations of Partial Differential Equations Occurring in Petroleum Engineering". Master of Science Thesis. Texas A&M University.
[7] Dass H. K. (2008). Advanced Engineering Mathematics. S. Chand.
[8] David L.P. (2006). Boundary Value Problems and Partial Differential Equations. Fourth edition.
[9] Erich M. (2012). Partial Differential Equations Lecture Notes. Department of Mathematics Leipzig University, Version October.
[10] Gabriel B.N, Omar E.O and Oscar A. (1994). "The Behaviour of Hyperbolic Heat Equations' Solutions near their Parabolic Limits". American Institute of Physics.
[11] Ilk, D. (2005). "Deconvolution of Variable Reservoir Performance Data Using B-Splines". Master of Science Thesis. Texas A&M University.
[12] John L. (1982). Well Testing. Texas A&M University.
[13] Johnston, J.L. and Lee W.J. (1991). "Interpreting Short-Term Buildup Tests from Low-Productivity Gas Wells Using Deconvolution". paper SPE 21503 presented at the Gas Technology Symposium, 22-24 January, Houston, Texas.
[14] Konstantin Z. (2016). Operational Approach and Solutions of Hyperbolic Heat Conduction Equations.
[15] Kreyszig E. (1999). Advanced Engineering Mathematics. John Wiley & Sons, INC.
[16] Levitan M. M. (2003). "Practical Application of Pressure-Rate Deconvolution to Analysis of Real Well Tests". Paper SPE 84290 presented at the SPE Annual Technical Conference and Exhibition, Denver, CO. 5-8 Oct.
[17] Levitan M. M., Crawford, G.E., Hardwick, A. (2004). "Practical Considerations for Pressure-Rate Deconvolution of Well Test Data". Paper SPE 90680 presented at the Annual Technical Conference and Exhibition, Houston, Texas, 26-29 Sep.
[18] Mahmood A. (2012). "Laplace Transform Deconvolution and its Application to Perturbation Solution of Non-Linear Diffusivity Equation". Ph.D Thesis. Colorado School of Mines.
[19] Martin K. Bessel Functions. Project for the Penn State - Göttingen Summer School on Number Theory.
[20] Michael M.L. (2003). Practical Application of Pressure-Rate Deconvolution to Analysis of Real Well Test.
[21] Morse P.M and Feshbach H. Methods of Theoretical Physics, Vol. I, Fesbach Publishing, LLC, Minneapolis, Minnesota, 1981.
[22] Muskat M. (1946). Flow of Homogeneous Fluids through Porous Media, 1st Ed., 2nd. Printing, J.W. Edwards, Inc., Ann Arbor, Michigan.
[23] Nnaemeka E. 2010. Petroleum Reservoir Engineering Practice. "Well Test Analysis: Deconvolution Concepts", pp. 525-536.

[24] Odeh A.S. and Jones, L.G. (1965). "Pressure Drawdown Analysis, Variable-Rate Case". JPT (Aug): 960-964; Trans., AIME 234. 73.
[25] Oroveanu T. and Pascal H (1959). "On the propagation of pressure waves in a liquid flowing through a porous medium". 4, 445-448.
[26] Roumboutsos A. and Stewart G. (1988). "A Direct Deconvolution or Convolution Algorithm for Well-Test Analysis". Paper SPE 18157 presented at the SPE Annual Technical Conference and Exhibition. Houston, TX. 2-5 Oct.
[27] Schroeter T.V. (2005). Deconvolution in Well Test Analysis. Symposium for Alain Gringarten.
[28] Stroud K.A. 2003. Advanced Engineering Mathematics (fourth edition). Palgrave Macmillan.
[29] [Van Everdingen A. F. and Hurst W. (1949). "The application of the Laplace transformation to flow problems in reservoirs", Transactions of the AIME 186, 305-324.