Laplace Transform Deconvolution and its Application to the Solution of the Hyperbolic Diffusivity Equation Under Wellbore Boundary Conditions

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Abstract: The deconvolution of variable rate and variable pressure data in well test analysis into, variable rate at constant pressure or variable pressure at constant rate is quite easily achieved in the Laplace domain operations. Issues involved with other spline-based methods are that: the piecewise linear function is discontinuous over its first derivative and as such will require many knots for better approximation of nonlinear trends, and the cubic spline has a higher tendency to oscillate around discontinuities. The major problems associated with the existing solution models of the parabolic diffusivity equation is the negligence of fluid density and possible inertia effects, and of course the inherent assumption of infinite speed of pressure propagation in the base transport equation, which implies inaccurate early times description. The first part of this work involves the analysis of variable rate and variable pressure data by the Laplace domain deconvolution process using the second order spline. The second order spline, when used with other functions can accurately transform sampled data into Laplace domain, other approaches found in the literature are used with the spline methods to handle discontinuities and noise in data, which make the method an algorithm that accurately transforms sampled data into Laplace domain. Secondly, is the extension of the convenience of the Laplace domain operations to solving the dimensionless radial flow hyperbolic diffusivity equation for infinite-acting systems. The hyperbolic diffusivity equation is a telegrapher’s model representing the reservoir fluid dynamics. Fluid inertia was taken into consideration in the base transport equation leading to the development of the hyperbolic diffusivity equation, thus, there is no assumption of infinite propagation speed of pressure disturbance. The Heaviside expansion method was further used to approximate the transformed variables before they are inverted back to time domain. Results were presented in figures to validate the proposed solution to the hyperbolic diffusivity equation and to compare with the solution offered to the parabolic diffusivity equation for various values of dimensionless (dummy) variable. The proposed solution to the hyperbolic flow equation captures the early time behaviour better, hence, better represents the reservoir system.

Keywords: Boundary Value Problem (IBVP), Initial conditions, Inner boundary conditions, Parabolic and Elliptic diffusivity equations, Volterra Integral Equation of the first kind.

1. Introduction

The deconvolution of variable rate and variable pressure data in Petroleum engineering, into variable pressure at constant rate or variable rate at constant pressure based on the Duhamel’s principle (the convolution integral for pressure and rate functions) requires the transformation of time dependent variables into Laplace domain, basically accomplished using the piecewise linear interpolation algorithm by fitting straight lines through successive knots, which has been found to be approximate for nonlinear trends in sampled data. The Laplace domain deconvolution method can be greatly extended in application if real time functions, which are only known as a table of the functions f(t) versus time (t) values, can be converted to Laplace space forms. The process of numerical Laplace transformation is relatively straight forward when the function f(t) is well behaved (Roumboutsos and Stewart, 1988). The piecewise linear approximation was introduced by Roumboutsos and Stewart in their work on direct deconvolution and convolution algorithm for well test analysis, to take Laplace transform of the data functions directly, on careful selection of the knots. Issues of oscillations due to noise in measured data are usually encountered and instabilities in numerical inversion processes because of discontinuities usually accompany the spline-based deconvolution methods. It is therefore required to have an adequate algorithm to transform the piecewise-continuous sampled data into the Laplace space and an appropriate numerical Laplace inversion algorithm capable of processing the exponential contributions caused by the tabulated data to exploit the potential of Laplace domain operations (Mahmood, 2012). To smoothen the deconvolved pressure response, an adaptive approach using a Gaussian and Epanechnikov kernel regression had been proposed. Also, the boundary mirroring approach was introduced by Mahmood to eliminate the effect of instabilities caused by discontinuities.

Most importantly, in this work, is the application of the famous Laplace integral transformation method to solving the
2. Mathematical basis of deconvolution

The Laplace transform $F(s)$ of a real function $f(t)$:

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt, \forall t > 0 \quad (1)$$

The Laplace transform of two convoluted functions $f(t)$ and $g(t)$ yields the product of the transforms of the two functions.

$$\mathcal{L}\{f(t) * g(t)\} = \mathcal{L}\{f(\tau)g(t - \tau)d\tau\} = F(s)G(s) \quad (2)$$

The Duhamel’s principle (Duhamel, 1833), used for solving pressure drop function, $\Delta p(r, t)$, at position $r$ and time $t$ due to variable rate $q_0(t)$, in the analysis of variable rate data in flow through porous media, is simply the application of eq. (2).

$$\Delta P(r, t) = \int_0^t q_D(\tau)\Delta P'_u(r, t - \tau)d\tau \quad (3)$$

The process of recovering the impulse or unit rate function $\Delta P_u(r, t)$ from measured variable rate response, $\Delta P(r, t)$, and variable rate, $q_0(t)$, is the reversal of the effect of convolution known as deconvolution. This implies that we are trying to solve a Volterra Integral Equation of the first kind.

By taking the Laplace transform of (3):

$$\Delta \bar{P}(s) = \bar{q}_D(s)\Delta \bar{P}_u(s) \quad (4)$$

But:

$$\mathcal{L}\{P'(t)\} = s\bar{P}(s) - P(0) \quad (5)$$

Therefore,

$$\mathcal{L}\{\Delta P'_u(t)\} = \Delta \bar{P}_u(s) = s\Delta \bar{P}_u(s) - \Delta P_u(0) \quad (6)$$

But at time, $t=0$, $\Delta P_u = 0$, therefore,

$$\Delta \bar{P}_u(s) = s\Delta \bar{P}_u(s) \quad (7)$$

Then, (4) becomes:

$$\Delta \bar{P}(s) = \bar{q}_D(s).s\Delta \bar{P}_u(s) \quad (8)$$

$$\Delta \bar{P}_u(s) = \frac{\Delta \bar{P}(s)}{s\bar{q}_D(s)} \quad (9)$$

On inversion back to time domain:

$$\Delta P_u(t) = \mathcal{L}^{-1}\left\{\frac{\Delta \bar{P}(s)}{s\bar{q}_D(s)}\right\} \quad (10)$$

This is the deconvolution process in which the constant rate response $\Delta P_u(t)$ has been recovered from the measured rate and pressure data. Once recovered, this constant rate response can then be analysed using the standard drawdown technique. The
application of (10) requires that measured field data, \( \Delta P(r, t) \) and \( q(t) \) be transformed to the Laplace transform domain using a proper approximating function and in addition, a suitable numerical Laplace transform inversion algorithm which effectively handles discontinuities and noise, to carry out the inversion process.

3. Second order spline based deconvolution

If \( S(t) \) is given over the range \( a \leq t \leq b \) with knots defined by:

\[
a = t_0 < t_1 < t_2 \ldots < t_n
\]  

(11)

The second order spline in each subinterval can be written as follows:

\[
S_i(t) = at_i^2 + bt_i + c, \quad i = 1, 2, \ldots, n
\]  

(12)

Where, \( S \) represents measured dependent variable data such as rate or pressure as a function of time, \( t \). The coefficient \( a, b \) and \( c \) are constants and different for each subinterval.

Second order spline function for each subinterval \( t_i-1 \leq t \leq t_i \) with values \( y_{i-1} \) and \( y_i \) at \( t_{i-1} \) and \( t_i \) respectively for pressure:

\[
P(t) = \frac{\ddot{m}}{2} (t_{i-1} - t)(t_i - t) + \frac{P_1(t - t_{i-1}) + P_{i-1}(t - t_{i})}{(t_i - t_{i-1})}
\]  

(13)

Where, \( \ddot{m} \) is the second derivative of the function. (13) can be rearranged as follows:

\[
P(t) = \frac{\ddot{m}t^2}{2} + \left[ \frac{P_i - P_{i-1}}{t_i - t_{i-1}} - \frac{\ddot{m}(t_i + t_{i-1})}{2} \right] t + \frac{\ddot{m}t_i t_{i-1} - P_{i-1}t_i - P_i t_{i-1}}{2(t_i - t_{i-1})} i = 2, 3, \ldots, n
\]  

(14)

Applying (1) to (14) over the interval \( t_1 < t < t_n \) for tabulated dependent variable data, such as pressure or production rate, gives:

\[
\mathcal{L}(P(t)) = \bar{P}(s) = \int_0^\infty P(t)e^{-st}dt = \int_0^{t_1} P_1(t)e^{-st}dt + \int_{t_1}^{t_n} P(t)e^{-st}dt + \int_{t_n}^\infty P(t)e^{-st}dt
\]  

(15)

But because the terms \( \int_0^{t_1} P(t)e^{-st}dt \) and \( \int_{t_n}^\infty P(t)e^{-st}dt \) are zero for a set of tabulated data within the interval \( t_1 < t < t_n \), they will therefore vanish. Substituting (14) into (15) yields the following Laplace transforms:

For the first term:

\[
\mathcal{L}\left\{ \frac{\ddot{m}t^2}{2} \right\} = \int_{t_1}^{t_n} \frac{\ddot{m}t^2}{2} e^{-st}dt
\]  

(16)

For the second term:

\[
\mathcal{L}\left\{ \frac{P_i - P_{i-1}}{t_i - t_{i-1}} - \frac{\ddot{m}(t_i + t_{i-1})}{2} \right\} t
\]  

(17)
For the third term:

\[ \mathcal{L} \left\{ \frac{\hat{m}_i t_{i-1}}{2} + \frac{P_{i-1} t_i - P_{i-1}}{t_i - t_{i-1}} \right\} \]

\[ = \int_{t_{i-1}}^{t_i} \left( \frac{\hat{m}_i t_i - P_{i-1} t_i - P_{i-1}}{t_i - t_{i-1}} \right) e^{-st} dt \]

\[ = \int_{t_{i-1}}^{t_i} \left( \frac{\hat{m}_i t_i - P_{i-1} t_i - P_{i-1}}{t_i - t_{i-1}} \right) e^{-st} dt + \]

\[ + \int_{t_{i-1}}^{t_{i-1}} \left( \frac{\hat{m}_i t_i - P_{i-1} t_i - P_{i-1}}{t_i - t_{i-1}} \right) e^{-st} dt + \]

\[ + \int_{t_{i-1}}^{t_{n-1}} \left( \frac{\hat{m}_i t_i - P_{i-1} t_i - P_{i-1}}{t_i - t_{i-1}} \right) e^{-st} dt \]

\[ \mathcal{L} \left\{ \frac{\hat{m}_i t_{i-1}}{2} + \frac{P_{i-1} t_i - P_{i-1}}{t_i - t_{i-1}} \right\} \]

\[ = \sum_{i=2}^{n} \left( \frac{\hat{m}_i t_{i-1}}{2} + \frac{P_{i-1} t_i - P_{i-1}}{t_i - t_{i-1}} \right) \left\{ - \frac{1}{s} e^{-st_i} \right\} \]

\[ = -e^{-st_{i-1}} \] (21)

For \( i = 2, 3, \ldots, n \) at \( t_i < t < t_n \).

**4. Problem formulation**

For a physical reservoir system of infinite radius \( R \) with a centered well of radius \( r \), the following simplifying assumptions are made:

1. The reservoir is homogeneous and isotropic with respect to permeability.
2. The formation is completely saturated with a single incompressible fluid.
3. Constant and pressure independent rock and fluid properties.
4. The well is completed across the entire formation thickness to assume a fully radial flow around the wellbore.
5. Negligible gravity forces.

**A. Mathematical model**

1) **Governing Equations**

**Continuity equation:**

\[ \nabla \cdot (\rho \vec{v}) = \frac{\partial (\rho v_x)}{\partial x} + \frac{\partial (\rho v_y)}{\partial y} + \frac{\partial (\rho v_z)}{\partial z} = \frac{\partial (\rho \phi)}{\partial t} \] (22)

**Transport Equation:**

By consolidating the idea of the generalised Darcy and Navier-Stokes equations, we arrived at a hydrodynamic equivalent to Newton’s second law of motion with the aim of creating a new set of transport equations that account for the effect of fluid density (inertia) and, also results to the generalized Darcy’s equation when the inertia tends to zero. (Oroveanu et al., 1959; Pascal, 1986):

\[ \frac{\rho \, d\vec{v}}{\partial t} = -\nabla P - \frac{\mu}{k} \vec{v} \] (23)

**Equation of state for slightly compressible fluid:**

\[ \rho = \rho_0 e^{(p-p_0)} \] (24)

**Formation compressibility:**

\[ c_f = \frac{1}{\rho \partial P} \] (25)

**Fluid compressibility:**

\[ c = \frac{1}{\rho \partial P} \] (26)

**Total compressibility:**

\[ c_t = c + c_f \] (27)

**Resulting Models:**

The flow models (hyperbolic diffusivity equations) result from the combination of equations (23) to (27):

**Linear:**

\[ \frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} + \frac{\partial^2 P}{\partial z^2} = \rho c_t \frac{\partial^2 P}{\partial t^2} + \frac{\mu \phi c_t}{k} \frac{\partial P}{\partial t} \] (28)

**Radial:**

\[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial P}{\partial r} \right) = \frac{1}{\alpha^2} \frac{\partial^2 P}{\partial t^2} + \frac{1}{\eta} \frac{\partial P}{\partial t} \] (29)

Where,

\[ \alpha^2 = \frac{1}{\rho c_t} \] (30)

\[ \eta = \frac{k}{\mu \phi c_t} \] (31)

In terms of dimensionless variables:

\[ \frac{1}{r_D} \frac{\partial}{\partial r_D} \left( r_D \frac{\partial P_D}{\partial r_D} \right) = \tau \frac{\partial^2 P_D}{\partial t_D^2} + \frac{\partial P_D}{\partial t_D} \] (32)
Where:
\[ P_D = \frac{2\pi k h (P_i - P)}{\mu q} \]  \hspace{1cm} (33)
\[ r_D = \frac{r}{r_w} \]  \hspace{1cm} (34)
\[ r_eD = \frac{r_e}{r_w} \]  \hspace{1cm} (35)
\[ t_D = \frac{\eta t}{r_w^2} \]  \hspace{1cm} (36)
\[ \eta = \frac{k}{\phi \mu c_i} \]  \hspace{1cm} (37)
\[ \tau = \frac{\eta}{\alpha \tau_w} \]  \hspace{1cm} (38)

5. Solution model

Equation (32) can also be expressed as:
\[ \frac{\partial^2 P_D}{\partial r_D^2} + \frac{1}{r_D} \frac{\partial P_D}{\partial r_D} = \tau^2 \frac{\partial^2 P_D}{\partial t_D^2} + \frac{\partial P_D}{\partial t_D} \]  \hspace{1cm} (39)

A. Laplace domain solution

I.C.s:
\[ P_D(r_D,t_D = 0) = 0 \]  \hspace{1cm} (40)
\[ \frac{\partial P_D}{\partial t_D}(r_D,t_D = 0) = 0 \]  \hspace{1cm} (41)

Taking the Laplace transform of (39):
\[ \frac{\partial^2 \tilde{P}_D}{\partial r_D^2}(r_D,s) + \frac{1}{r_D} \frac{\partial \tilde{P}_D}{\partial r_D}(r_D,s) \]
\[ = \tau^2 \left[ s^2 \tilde{P}_D(r_D,t = 0) - s P_D(r_D,t = 0) \right] \]
\[ - \frac{\partial \tilde{P}_D}{\partial t_D}(r_D,t = 0) \]
\[ - \tilde{P}_D(r_D,0) \]  \hspace{1cm} (42)

Inserting the initial conditions:
\[ \frac{\partial^2 \tilde{P}_D}{\partial r_D^2}(r_D,s) + \frac{1}{r_D} \frac{\partial \tilde{P}_D}{\partial r_D}(r_D,s) - (\tau^2 s^2 + s) \tilde{P}_D(r_D,s) \]
\[ = 0 \]  \hspace{1cm} (43)

Transformation to modified Bessel equation
\[ z^2 \frac{\partial^2 P_D}{\partial z^2}(r_D,s) + z \frac{\partial P_D}{\partial z}(r_D,s) - z^2 P_D(r_D,s) \]
\[ = 0 \]  \hspace{1cm} (44)

Where:
\[ z^2 = r_D(\tau^2 s^2 + s) \]  \hspace{1cm} (45)

1) Solution for terminal rate inner boundary condition
\[ P_D = \frac{2\pi k h (P_i - P)}{\mu q} \]  \hspace{1cm} (46)

B.C.:
\[ r_D \frac{\partial P_D}{\partial r_D} = -1, \quad \text{when } r_D = 1, \forall t_D > 0 \]  \hspace{1cm} (47)

General solution to (44):
\[ \tilde{P}_D(r_D,s) = A \tilde{I}_0 \left( r_D \sqrt{\tau^2 s^2 + s} \right) \]
\[ + B K_0 \left( r_D \sqrt{\tau^2 s^2 + s} \right) \]  \hspace{1cm} (48)

A = 0, for pressure to remain finite as \( r_D \to \infty \)

From the boundary condition:
\[ r_D \frac{\partial \tilde{P}_D}{\partial r_D} = -\frac{1}{s} \frac{\partial \tilde{P}_D}{\partial t_D}, \quad \text{when } r_D = 1 \]  \hspace{1cm} (49)

Inserting the boundary condition and solving for the constant:
\[ \tilde{P}_D(1,s) = \frac{K_0(\sqrt{\tau^2 s^2 + s})}{s \sqrt{\tau^2 s^2 + s} + s K_1(\sqrt{\tau^2 s^2 + s})} \]  \hspace{1cm} (50)

At the wellbore, \( r_D = 1 \), therefore,
\[ \tilde{P}_D(1,s) = \frac{K_0(\sqrt{\tau^2 s^2 + s})}{s \sqrt{\tau^2 s^2 + s} + s K_1(\sqrt{\tau^2 s^2 + s})} \]  \hspace{1cm} (51)

For very small \( t_D \):
\[ K_0(z) = K_1(z) = \frac{\pi}{\sqrt{2z}} e^{-z} \]  \hspace{1cm} (52)

Therefore,
\[ \tilde{P}_D(1,s) = \frac{1}{s \sqrt{\tau^2 s^2 + s}} \]  \hspace{1cm} (53)

By Heaviside expansion and inverse transform:
\[ P_D(1,t_D) \approx \frac{t_D}{\tau} \frac{2}{\Gamma(1/2)} + \frac{3t_D^{-3/2}}{8} \frac{\pi^2}{16} + \frac{5t_D^{-1/2}}{2} \frac{\Gamma(5/2)}{3} + \frac{105t_D^{-7/2}}{384} \frac{\Gamma(3/2)}{\Gamma(1/2)} + \cdots \]  \hspace{1cm} (54)
\[ P_D(t_D) \approx 2 \sqrt{\frac{t_D}{\pi}} - \frac{\tau^2}{2\sqrt{\pi t_D}} - \frac{3\tau^4}{16\sqrt{\pi t_D}^{3/2}} - \frac{15\tau^6}{64\sqrt{\pi t_D}^{5/2}} - \ldots \]

For larger \( t_D \):
\[ K_0(z) \approx \frac{1}{z} \ln \left( \frac{4}{e^{2\tau} z^2} \right) \]
\[ K_1(z) \approx \frac{1}{z} \]

Therefore,
\[ P_D(1,s) = \frac{1}{2} \ln \left( \frac{4}{e^{2\tau} s^2 + s} \right) \]

By Heaviside expansion and inversion back to time domain:
\[ P_D(1,t_D) = \ln 2 - \gamma - \frac{1}{2}(-\gamma - \ln t_D) \]
\[ P_D(1,t_D) = \ln 2 - \sqrt{t_D} - \frac{1}{2} \gamma \]

2) Solution for terminal pressure inner boundary condition
\[ P_D = P_{1} - P \]

B.C:
\[ P_D(r_D = 1, t_D) = 1 \] \( \forall \ t_D > 0 \)

Therefore,
\[ P_D(1,s) = \frac{1}{s} \]

Solution:
\[ P_D(r_D,s) = \frac{K_0(r_D\sqrt{s^2} + s)}{s K_0(\sqrt{s^2} + s)} \]

\[ \tilde{q}_D = -r_D \frac{\partial P_D}{\partial r_D} \]

From (63),
\[ \frac{\partial P_D}{\partial r_D}(r_D,s) = -\sqrt{s^2} + s \frac{K_0(r_D\sqrt{s^2} + s)}{s K_0(\sqrt{s^2} + s)} \]

Therefore,
\[ \tilde{q}_D(r_D,s) = \frac{r_D\sqrt{s^2} + s}{s K_0(\sqrt{s^2} + s)} \]

At the inner boundary (wellbore):
\[ \bar{q}_D(1,s) = \frac{\sqrt{s^2} + s}{s} \frac{K_0(\sqrt{s^2} + s)}{s K_0(\sqrt{s^2} + s)} \]

For very small \( t_D \), (68) reduces to:
\[ \bar{q}_D(1,s) = \frac{\sqrt{s^2} + s}{s} \]

But,
\[ Q_D(1,t_D) = \int_0^{t_D} \tilde{q}_D(1,t_D) \, dt \]

Taking the Laplace transform of (70):
\[ \bar{Q}_D(1,s) = \frac{\bar{q}_D(1,s)}{s} \]

Therefore,
\[ \bar{Q}_D(1,s) = \frac{\sqrt{s^2} + s}{s^2} = \frac{\sqrt{s^2} + 1}{s^{3/2}} \]

By Heaviside expansion and inverse transform to time domain:
\[ Q_{DC}(1,t_D) \approx \frac{t_D^{1/2}}{\Gamma \left( \frac{1}{2} + 1 \right)} + \frac{t_D^{-1/2} \tau^2}{2 \Gamma \left( -\frac{1}{2} + 1 \right)} - \frac{8 \tau^4}{128 \Gamma \left( -\frac{1}{2} + 1 \right)} + \frac{16 \tau^6}{128 \Gamma \left( -\frac{1}{2} + 1 \right)} - \ldots \]

As seen before, solution to the hyperbolic diffusivity equation for larger time approximation is expected to yield the same result as that of its parabolic counterpart. This implies that the solution model for the parabolic equation converges to that of its hyperbolic counterpart after some time from the beginning of flow.

6. Results

The dimensionless pressure solutions to the famous parabolic diffusivity equation (the case for \( \tau = 0 \)) can be found in the work of Van A.F and Hurst W (1949), which is compared with the solutions obtained in this work. Furthermore, log-log plots of the \( P_D \) alongside \( P_D(\text{Der}) \) versus \( t_D \) for the various values of \( \tau \) are shown for more illustration of the distinction.
7. Discussion

The solutions obtained in this work yield the same results for the larger times approximation as those of the work of Van Everdingen and Hurst (1949). These solutions are useful for pressure profile analysis which is vital for production optimization in the oil and gas industry.

Also, the derivative plots show the stabilized dimensionless pressure of 0.5 which we wish to maintain. This is the region which indicates the ideal fully radial flow for the infinite acting system that is producing clean oil.

As presented in fig. 1, it is observed that the inherent
assumption of infinite speed of pressure propagation through the fluid in Darcy’s Law due to the negligence of possible inertia effect will have some effects on the results obtained. As clearly demonstrated, the solutions to the radial flow parabolic diffusivity equation will only converge to that of its hyperbolic counterpart after sufficient time. It is therefore of the essence to model with the hyperbolic diffusivity equation if the description of the early time flow behaviour of the reservoir system is vital in any reservoir engineering analysis.

8. Conclusion

The extension of the convenience of the Laplace domain deconvolution to solving the hyperbolic diffusivity equation, which is an extension on the classical work of Van Everdingen and Hurst (1949), clearly shows that the model (hyperbolic diffusivity equation) can accurately model the reservoir flow behaviour and better represents the early times flow nature of the reservoir system.

Nomenclature

\[ \mathcal{L} = \text{Laplace operator}. \]
\[ s = \text{Laplace transform parameter}. \]
\[ \forall = \text{for all}. \]
\[ \Delta P(r, t) = \text{pressure drop function}. \]
\[ \Delta P_u = \text{impulse (unit or constant rate) function drop}. \]
\[ \Delta P'_u = \text{derivative of the impulse function drop}. \]
\[ r_w = \text{wellbore radius}. \]
\[ \mu = \text{fluid viscosity}. \]
\[ \phi = \text{formation porosity}. \]
\[ k = \text{formation permeability}. \]
\[ h = \text{pay thickness}. \]
\[ r_D = \text{dimensionless radius}. \]
\[ t_p = \text{dimensionless time}. \]
\[ \tau = \text{dummy variable}. \]
\[ q_0 = \text{dimensionless flow rate}. \]
\[ Q_0 = \text{dimensionless cumulative production}. \]
\[ p_0 = \text{dimensionless pressure}. \]
\[ q'_0 = \text{transformed dimensionless rate}. \]
\[ Q'_0 = \text{transformed dimensionless cumulative production}. \]
\[ p'_0 = \text{transformed dimensionless pressure}. \]
\[ \nabla = \text{div operator}. \]
\[ \vec{v} = \text{directional velocity}. \]
\[ \rho = \text{density}. \]
\[ c_f = \text{formation compressibility}. \]
\[ c = \text{fluid compressibility}. \]
\[ c_t = \text{total compressibility}. \]
\[ \eta = \text{hydraulic diffusivity}. \]
\[ \infty = \text{speed of sound through fluid}. \]
\[ A, B = \text{constants in the general solution to the modified Bessel equation}. \]
\[ I_0 = \text{zeroth order modified Bessel function of the first kind}. \]
\[ K_0 = \text{zeroth order modified Bessel function of the second kind}. \]
\[ K_1 = \text{first order modified Bessel function of the second kind}. \]

\[ e = \text{exponentiation}. \]
\[ \infty = \text{infinity}. \]
\[ \pi \approx 3.142 \]
\[ \gamma = \text{Euler’s constant} \approx 0.5772. \]
\[ \partial = \text{partial differential}. \]
\[ \int = \text{integral sign}. \]

References