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Existence of Solutions for Nonlinear Impulsive Fractional Differential Equation with Periodic Boundary Conditions

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***Abstract*: This paper is we investigate theoretical development in fractional differential equation with periodic boundary condition by using monotone iterative method.**

***Keywords*: Impulsive, Riemann- Liouville sequential fractional derivative, periodic boundary value problem.**

1. Introduction

In introductory course on fractional differential equations,

$$\mathcal{D}^{2\alpha}v(t) = f(t, v, \mathcal{D}^{\alpha}u), \quad t\epsilon(0,1]\backslash\{t_{1,\dots,}t_m\}, 0 < \alpha \leq 1, \quad (1)$$

$$\lim_{t\to 0^+} t^{1-\alpha}\, v(t) = v(1)\ , \quad \lim_{t\to 0^+} t^{1-\alpha}\mathcal{D}^{\alpha}\, u(t) = \mathcal{D}^{\alpha}u(1), \quad (2)$$

$$\lim_{t\to t_j^+}(t - t_j)^{1-\alpha}\left(v(t) - v(t_j)\right) = I_j\left(v(t_j)\right), \quad (3)$$

$$\lim_{t\to t_j^+}(t - t_j)^{1-\alpha}\left(\mathcal{D}^{\alpha}v(t) - \mathcal{D}^{\alpha}v(t_j)\right) = \bar{I}_j\left(v(t_j)\right), \quad (4)$$

Where,

$$\mathcal{D}^{\alpha}v(t) = \left(0^{\mathcal{D}_t^{\alpha}v}\right)(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-\tau)^{-\alpha}\, v(\tau)\, d\tau$$

is the standard Riemann–Liouville fractional derivative, $\mathcal{D}^{2\alpha} = \mathcal{D}^{\alpha}(\mathcal{D}^{\alpha}v)$ is the sequential Riemann–Liouville fractional derivative .

$0 < t1 < t2 < \cdot\cdot\cdot < t_m < 1, I_j, \bar{I}_j \in C(R, R)\ (j = 1,2,\dots,m)\ f$ is continuous at every point $(t, u, v) \in [0, 1] \times R \times R$,

Differential equation with fractional order have recently proved valuable tools in the modeling of many phenomena in various fields of science and engineering. Recently, many researchers have paid attention to existence result of solution of the initial value problem and boundary value problem for fractional differential equations. For example, Belmekki et al. investigated the existence and uniqueness of solution of the following fractional differential equation with periodic boundary value condition

$$\mathcal{D}^{\delta}v - \lambda v(t) = f\left(t, v(t)\right), t \in (0, 1], \quad 0 < \ \delta < 1, \quad (5)$$

$$\lim_{t\to 0^+} t^{1-\delta}v(t) = \ v(1), \quad (6)$$

$$\mathcal{D}_{0+}^{2\alpha}y(x) = f(x, y, \mathcal{D}_{0+}^{\alpha}y), \quad x \in (0, T], \quad (7)$$

$$x^{1-\alpha}y(x)|_{x=0} = y_0, x^{1-\alpha}(\mathcal{D}_{0+}^{\alpha}y)(x) = y(1) \quad (8)$$

by using monotone iterative method, where $\mathcal{D}_{0+}^{\alpha} = \mathcal{D}^{\alpha}$and $\mathcal{D}_{0+}^{2\alpha} = \mathcal{D}^{2\alpha}$are as mentioned above.

A. *Theorem*

The linear impulsive boundary value problem

$$\mathcal{D}^{2\alpha}v(t) + p\mathcal{D}^{\alpha}v(t) + qv(t) = \sigma(t), t\epsilon(0,1]\backslash\{t_1, \dots, t_m\} \quad (9)$$

$$\lim_{t\to 0^+} t^{1-\alpha}v(t) = v(1), \qquad \lim_{t\to 0^+} t^{1-\alpha}\mathcal{D}^{\alpha}v(t) = \ \mathcal{D}^{\alpha}v(1), \quad (10)$$

$$\lim_{t\to t_j^+}(t - t_j)^{1-\alpha}\left(v(t) - v(t_j)\right) = a_{j,} \qquad j = 1, \dots, m, \quad (11)$$

$$\lim_{t\to t_j^+}(t - t_j)^{1-\alpha}\left(\mathcal{D}^{\alpha}v(t) - \mathcal{D}^{\alpha}v(t_j)\right) = b_{j,}\ \ j = 1, \dots, m, \quad (12)$$

Where $p, q, a_j, b_j \in R$ are constants with $p, q > 0$ and $p^2 > 4q$ and $\sigma\epsilon C[0,1]$, has the following representation of solutions

$$v(t) = \int_0^1 G_{\lambda,\alpha}(t,s)\int_0^1 G_{\lambda_{1,\alpha}}(s,\tau)\sigma(\tau)\,d\tau + \sum_{j=1}^m \Gamma(\alpha)\left(b_j - \lambda_2 a_j\right)\int_0^1 G_{\lambda_{2,\alpha}}(t,s)G_{\lambda_{1,\alpha}}\left(s, t_j\right)ds + \sum_{j=1}^m \Gamma(\alpha)G_{\lambda_{2,\alpha}}\left(t, t_j\right)a_{j,} \quad (13)$$

where $G_{\lambda_{i,\alpha}}(t,s)(i = 1,2)$

$$\lambda_1 = \frac{-p+\sqrt{p^2-4q}}{2} < 0, \qquad \lambda_1 = \frac{-p+\sqrt{p^2-4q}}{2} \quad (14)$$

B. *Proof*

Let $(\mathcal{D}^{\alpha} - \lambda_2)u(t) = x(t), t\epsilon(0,1]\backslash\{t_1, \dots, t_m\}$.

Then the problem $(9) - (12)$is equivalent to

$$(\mathcal{D}^{\alpha} - \lambda_2)v(t) = x(t)\ ,\ \ t\epsilon(0,1]\backslash\{t_1, \dots, t_m\}, \quad (15)$$

$$\lim_{t\to 0^+} t^{1-\alpha}\, v(t) = \ v(1) \quad (16)$$

$$\lim_{t\to t_j^+}(t - t_j)^{1-\alpha}\left(u(t) - u(t_j)\right) = a_j \quad (17)$$

And $(\mathcal{D}^{\alpha}-\lambda_1)x(t)=\sigma(t)\ , \quad t\epsilon(0,1]\backslash\{t_1,\dots,t_m\},$ (18)

$$\lim_{t\to 0^+} t^{1-\alpha}\,x(t)=x(1), \quad (19)$$

$$\lim_{t\to t_j^+}(t-t_j)^{1-\alpha}\left(x(t)-x(t_j)\right)=b_j-\lambda_2 a_j \quad (20)$$

$k=0$, we obtain that BVPs$(15)-(17)$and $(18)-(20)$have the following representation of solutions

$$v(t)=\int_0^1 G_{\lambda_{2,\alpha}}(t,s)x(s)\,ds+\sum_{j=1}^{m}\Gamma(\alpha)\,G_{\lambda_{2,\alpha}}(t,t_j)a_j \quad (21)$$

$$x(t)=\int_0^1 G_{\lambda_{1,\alpha}}(t,s)\sigma(s)\,ds+\sum_{j=1}^{m}\Gamma(\alpha)\,G_{\lambda_{1,\alpha}}(t,t_j)(b_j-\lambda_2 a_j) \quad (22)$$

Respectively, Substituting(22)into (21), we get

$$v(t)=\int_0^1 G_{\lambda,\alpha}(t,s)\int_0^1 G_{\lambda_{1,\alpha}}(s,\tau)\sigma(\tau)\,d\tau+\sum_{j=1}^{m}\Gamma(\alpha)(b_j-\lambda_2 a_j)\int_0^1 G_{\lambda_{2,\alpha}}(t,s)G_{\lambda_{1,\alpha}}(s,t_j)\,ds+\sum_{j=1}^{m}\Gamma(\alpha)G_{\lambda_{2,\alpha}}(t,t_j)a_j,$$

Hence proved.

2. Main result

Let $v_0,w_0\in PC_{1-\alpha}[0,1]$. $v_0$is called a lower solution of the problem$(1)-(4)$if it satisfies,

$$\mathcal{D}^{2\alpha}v_0(t)\le f(t,v_0,\mathcal{D}^{\alpha}v_0),\in(0,1]\backslash\{t_1,\dots,t_m\}, \quad (23)$$

$$\lim_{t\to 0^+}t^{1-\alpha}v_0(t)\le v_0(1),\qquad \lim_{t\to 0^+}t^{1-\alpha}\mathcal{D}^{\alpha}v_0(t)\le\mathcal{D}^{\alpha}v_0(1), \quad (24)$$

$$\lim_{t\to t_j^+}(t-t_j)^{1-\alpha}\left(v_0(t)-v_0(t_j)\right)\le I_j\left(v_0(t_j)\right), \quad (25)$$

$$\lim_{t\to t_j^+}(t-t_j)^{1-\alpha}\left(\mathcal{D}^{\alpha}v_0(t)-\mathcal{D}^{\alpha}v_0(t_j)\right)\le\bar{I}_j\left(v_0(t_j)\right) \quad (26)$$

And $w_0$is called an upper solution of the problem $(1)-(4)$ if it satisfies

$$\mathcal{D}^{2\alpha}w_0(t)\le f(t,w_0,\mathcal{D}^{\alpha}w_0),\qquad t\in(0,1]\backslash\{t_1,\dots,t_m\}, \quad (27)$$

$$\lim_{t\to 0^+}t^{1-\alpha}w_0(t)\le w_0(1),\ \lim_{t\to 0^+}t^{1-\alpha}\mathcal{D}^{\alpha}w_0(t)\ge\mathcal{D}^{\alpha}w_0(1), \quad (28)$$

$$\lim_{t\to t_j^+}(t-t_j)^{1-\alpha}\left(w_0(t)-w_0(t_j)\right)\le I_j\left(w_0(t_j)\right), \quad (29)$$

$$\lim_{t\to t_j^+}(t-t_j)^{1-\alpha}\left(\mathcal{D}^{\alpha}w_0(t)-\mathcal{D}^{\alpha}w_0(t_j)\right)\le\bar{I}_j\left(w_0(t_j)\right) \quad (30)$$

In the following, we assume that

$$\begin{cases} v_0(t)\le w_0(t), t\epsilon(0,1]:\ \lim_{t\to 0^+}t^{1-\alpha}v_0(t)\le\lim_{t\to 0^+}t^{1-\alpha}w_0(t),\\ \lim_{t\to 0^+}t^{1-\alpha}\mathcal{D}^{\alpha}v_0(t)\le\lim_{t\to 0^+}t^{1-\alpha}\mathcal{D}^{\alpha}w_0(t),\end{cases} \quad (31)$$

and define the order interval in space $PC_{1-\alpha}^{\alpha}[0,1]$:

$$[v_0,w_0]=\{u\in PC_{1-\alpha}^{\alpha}[0,1]: v_0(t)\le u(t)\le w_0(t), t\in(0,1],$$

$$\lim_{t\to 0^+}t^{1-\alpha}v_0(t)\le\lim_{t\to 0^+}t^{1-\alpha}u(t)\le\lim_{t\to 0^+}t^{1-\alpha}w_0(t),$$

$$\lim_{t\to 0^+}t^{1-\alpha}v_0(t)\le\lim_{t\to 0^+}t^{1-\alpha}\mathcal{D}^{\alpha}u(t)\le\lim_{t\to 0^+}t^{1-\alpha}\mathcal{D}^{\alpha}w_0(t)\}$$

Let

$$M_1(T)\coloneqq\mathcal{D}^{\alpha}v_0(t)+\lambda_2\left(w_0(t)-v_0(t)\right),\ M_2(T)\coloneqq\mathcal{D}^{\alpha}w_0(t)+\lambda_2\left(w_0(t)-v_0(t)\right),$$

For convenience, we shall assume that $f$satisfies the following conditions:

(H_1) There exist constants $p,q>0$with $p^2>4q$ such that

$$f(t,w_0,\mathcal{D}^{\alpha}w_0)-f(t,w_0,\mathcal{D}^{\alpha}v_0)\ge -p(\mathcal{D}^{\alpha}w_0-\mathcal{D}^{\alpha}v_0)-q(w_0-v_0),$$

Where $t\in(0,1]\backslash\{t_1,\dots,t_m\}, v_0,w_0\in PC_{1-\alpha}^{\alpha}[0,1]$are lower and upper solutions of problem$(1)-(4)$;

(H_2) There exist constants $p,q>0$ with p^2-4q such that $f(t,x_2,y_2)-f(t,x_1,y_1)\ge -p(y_2-y_1)-q(x_2-x_1),$

Where, $t\in(0,1]\backslash\{t_1,\dots,t_m\}, v_0(t)\le x_1\le x_2\le w_0(t), M_1(t)\le y_i\le M_2(t), i=1,2;$

$(H_3)I_j,\bar{I}_j\in C(R,R), I_j(y)\ge I_j(x)$ and $I_j(y)\ge\bar{I}_j(x)$, $\forall v_0(t_j)\le x\le y\le w_0(t_j), j=1,2\dots,m.$

A. *Example*

suppose that $(H_1)$and (H_3) hold. Then

$$\mathcal{D}^{\alpha}\left(w_0(t)-v_0(t)\right)-\lambda_2\left(w_0(t)-v_0(t)\right)\ge 0,\quad t\in(0,1].$$

B. *Proof*

Let

$$y(t)=\mathcal{D}^{\alpha}\left(w_0(t)-v_0(t)\right)-\lambda_2\left(w_0(t)-v_0(t)\right), t\in(0,1].$$

Then by $(H_1)$and (H_3),we have

$$\mathcal{D}^{\alpha}y(t)-\lambda_1 y(t)=\mathcal{D}^{2\alpha}\left(w_0(t)-v_0(t)\right)+p\mathcal{D}^{\alpha}\left(w_0(t)-v_0(t)\right)+q\left(w_0(t)-v_0(t)\right)$$
$$\ge f(t,w_0,\mathcal{D}^{\alpha}w_0)-f(t,v_0,\mathcal{D}^{\alpha}v_0)+p\mathcal{D}^{\alpha}\left(w_0(t)-v_0(t)\right)-q\left(w_0(t)-v_0(t)\right)\ge 0$$

$$\lim_{t\to 0^+} t^{1-\alpha} y(t) - y(1) = \lim_{t\to 0^+} t^{1-\alpha}\big(\mathcal{D}^\alpha w_0(t) - \mathcal{D}^\alpha v_0(t)\big) - \mathcal{D}^\alpha\big(w_0(1) - v_0(1)\big)$$

$$-\lambda_2 \lim_{t\to 0^+} t^{1-\alpha}\big(w_0(t) - v_0(t)\big) + \lambda_2\big(w_0(1) - v_0(1)\big) \geq 0,$$

and

$$\lim_{t\to t_j^+}\left(t - t_j\right)^{1-\alpha}\left(y(t) - y(t_j)\right) \geq \bar{I}_j\left(w_0(t_j)\right) - \bar{I}_j\left(v_0(t_j)\right) - \lambda_2\left[I_j\left(w_0(t_j)\right) - \left(v_0(t_j)\right)\right] \geq 0, j = 1,2\ldots,m.$$

Have $y(t) \geq 0$ for $t \in (0,1]$.

Hence the proof.

3. Conclusion

This paper presented existence of solutions for nonlinear impulsive fractional differential equation with periodic boundary conditions.

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