Existence of Solutions for Nonlinear Impulsive Fractional Differential Equation with Periodic Boundary Conditions

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Abstract: This paper is we investigate theoretical development in fractional differential equation with periodic boundary condition by using monotone iterative method.

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1. Introduction

In introductory course on fractional differential equations, 
\[ D^\alpha \nu(t) = f(t, \nu, D^\beta u), \quad te(0,1] \setminus \{t_1, t_m\}, 0 < \alpha \leq 1, \]  
(1)

\[ \lim_{t \to 0^+} t^{1-\alpha} \nu(t) = \nu(1), \quad \lim_{t \to 0^+} t^{1-\alpha} D^\alpha u(t) = D^\alpha u(1), \]  
(2)

\[ \lim_{t \to t_\nu} (t - t_\nu)^{1-\alpha} \left( \nu(t) - \nu(t_\nu) \right) = I_\nu \left( \nu(t_\nu) \right), \]  
(3)

\[ \lim_{t \to t_\nu} (t - t_\nu)^{1-\alpha} \left( D^\alpha \nu(t) - D^\alpha \nu(t_\nu) \right) = I_\nu \left( \nu(t_\nu) \right), \]  
(4)

Where, 
\[ D^\alpha \nu(t) = \left( \frac{d^\alpha}{dt^\alpha} \right) \nu(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\tau)^{-\alpha} \nu(\tau) d\tau \]  
is the standard Riemann–Liouville fractional derivative, \( D^{2\alpha} = D^\alpha (D^\alpha \nu) \) is the sequential Riemann–Liouville fractional derivative.

0 < t_1 < t_2 < \cdots < t_m < 1, I_{j}, \overline{I}_{j} \in C(R, R) (j = 1, 2, \ldots, m) \) \( \nu \) is continuous at every point \( (t, u, \nu) \in [0, 1] \times R \times R \),

Differential equation with fractional order have recently proved valuable tools in the modeling of many phenomena in various fields of science and engineering. Recently, many researchers have paid attention to existence result of solution of the initial value problem and boundary value problem for fractional differential equations. For example, Belmekki et al. investigated the existence and uniqueness of solution of the following fractional differential equation with periodic boundary value condition

\[ D^\delta \nu - \lambda \nu(t) = f(t, \nu(t)), t \in (0,1], 0 < \delta < 1, \]  
(5)

\[ \lim_{t \to 0^+} t^{1-\delta} \nu(t) = \nu(1), \]  
(6)

\[ D^{2\alpha} \nu(x) = f(x, y, D^\alpha y), \quad x \in (0, T], \]  
(7)

\[ x^{1-\alpha} y(x) \big|_{x=0} = y_0, x^{1-\alpha} (D^\alpha y)(x) = y(1) \]  
(8)

by using monotone iterative method, where \( D_0^\alpha = D^\alpha \) and \( D_0^{2\alpha} = D^{2\alpha} \) are as mentioned above.

A. Theorem

The linear impulsive boundary value problem
\[ D^{2\alpha} \nu(t) + p D^\alpha \nu(t) + q \nu(t) = \sigma(t), te(0,1] \setminus \{t_1, t_m, \ldots, t_m \} \]  
(9)

\[ \lim_{t \to 0^+} t^{1-\alpha} \nu(t) = \nu(1), \quad \lim_{t \to 0^+} t^{1-\alpha} D^\alpha \nu(t) = D^\alpha \nu(1), \]  
(10)

\[ \lim_{t \to t_\nu} (t - t_\nu)^{1-\alpha} \left( \nu(t) - \nu(t_\nu) \right) = a_j, \quad j = 1, \ldots, m, \]  
(11)

\[ \lim_{t \to t_\nu} (t - t_\nu)^{1-\alpha} \left( D^\alpha \nu(t) - D^\alpha \nu(t_\nu) \right) = b_j, \quad j = 1, \ldots, m, \]  
(12)

Where \( p, q, a_j, b_j \in R \) are constants with \( p, q > 0 \) and \( p^2 > 4q \) and \( \sigma \in [0,1] \), has the following representation of solutions
\[ \nu(t) = \int_0^t G_{\alpha, \lambda_\nu}(t, s) \int_0^s \int_0^t G_{\alpha, \lambda_\lambda}(s, \tau) \sigma(\tau) d\tau + \sum_{j=1}^m \Gamma(\alpha)(h_j - \lambda_j a_j) \int_0^t G_{\alpha, \lambda_{2, a}}(t, s) G_{\alpha, \lambda_{2, a}}(s, t_\nu) ds + \sum_{j=1}^m \Gamma(\alpha) G_{\alpha, \lambda_{2, a}}(t, t_\nu) a_j, \]  
(13)

where \( G_{\lambda_{k, a}}(t, s), i = 1, 2 \).

\[ \lambda_1 = \frac{p+\sqrt{p^2-4q}}{2} < 0, \quad \lambda_4 = \frac{-p+\sqrt{p^2-4q}}{2} \]  
(14)

B. Proof

Let \( (D^\alpha - \lambda_2) \nu(t) = x(t), te(0,1] \setminus \{t_1, \ldots, t_m \}. \)

Then the problem \( (9) - (12) \) is equivalent to
\[ (D^\alpha - \lambda_2) v(t) = x(t), \quad te(0,1] \setminus \{t_1, \ldots, t_m \}, \]  
(15)

\[ \lim_{t \to 0^+} t^{1-\alpha} v(t) = v(1) \]  
(16)

\[ \lim_{t \to t_\nu} (t - t_\nu)^{1-\alpha} (u(t) - u(t_\nu)) = a_j \]  
(17)
And \((D^\alpha - \lambda_j)x(t) = \sigma(t), \quad t \in (0, 1] \setminus \{t_1, \ldots, t_m\}\),
\[
\lim_{t \to 0^+} t^{1-\alpha} x(t) = x(1),
\]
\[
\lim_{t \to t_j^-} (t - t_j)^{1-\alpha} (x(t) - x(t_j)) = b_j - \lambda_2 a_j
\]
\[
\text{respectively, Substituting(22) into (21), we get}
\]
\[
v(t) = \int_0^1 G_{\lambda_2, \alpha}(t, s) \sigma(s) \, ds + \sum_{j=1}^m \Gamma(\alpha) G_{\lambda_1, \alpha}(t, t_j)(b_j - \lambda_2 a_j)
\]
\[
\text{Hence proved.}
\]
\section{Main result}

Let \(v_0, v_0 \in PC_{\alpha-}[0, 1]\). \(v_0\) is called a lower solution of the problem (1) – (4) if it satisfies,
\[
D^{2\alpha}v_0(t) \leq f(t, v_0, D^\alpha v_0), t \in (0, 1] \setminus \{t_1, \ldots, t_m\}, \quad v_0(0) \leq 0,
\]
\[
\lim_{t \to t_j^-} (t - t_j)^{1-\alpha} (v_0(t) - v_0(t_j)) \leq l_j(v_0(t_j)),
\]
\[
\text{and} \quad w_0(t) \text{ is called an upper solution of the problem (1) – (4) if it satisfies}
\]
\[
D^{2\alpha}w_0(t) \leq f(t, w_0, D^\alpha w_0), t \in (0, 1] \setminus \{t_1, \ldots, t_m\},
\]
\[
\lim_{t \to t_j^-} (t - t_j)^{1-\alpha} (w_0(t) - w_0(t_j)) \leq l_j(w_0(t_j)),
\]
\[
\text{In the following, we assume that}
\]
\[
\begin{align*}
& v_0(t) \leq v_0(t), t \in (0, 1],
& \lim_{t \to 0^+} t^{1-\alpha} v_0(t) \leq \lim_{t \to 0^+} t^{1-\alpha} w_0(t),
& \lim_{t \to 0^+} t^{1-\alpha} D^\alpha v_0(t) \leq \lim_{t \to 0^+} t^{1-\alpha} D^\alpha w_0(t),
\end{align*}
\]
\[
\text{and define the order interval in space } PC_{\alpha-}[0, 1]:
\]
\[
[v_0, w_0] = \{ u \in PC_{\alpha-}[0, 1]: v_0(t) \leq u(t) \leq w_0(t), t \in (0, 1],
\]
\[
\lim_{t \to 0^+} t^{1-\alpha} v_0(t) \leq \lim_{t \to 0^+} t^{1-\alpha} w_0(t),
\]
\[
\lim_{t \to 0^+} t^{1-\alpha} D^\alpha v_0(t) \leq \lim_{t \to 0^+} t^{1-\alpha} D^\alpha w_0(t)
\]
\[
\text{Let}
\]
\[
M_1(T) := D^\alpha w_0(t) + \lambda_2 (w_0(t) - v_0(t)), \quad M_2(T) := D^\alpha w_0(t) + \lambda_2 (w_0(t) - v_0(t)),
\]
\[
\text{For convenience, we shall assume that } f \text{ satisfies the following conditions:}
\]
\[
(H_1) \text{ There exist constants } p, q > 0 \text{ with } p^2 > 4q \text{ such that}
\]
\[
f(t, v_0, D^\alpha v_0) - f(t, w_0, D^\alpha w_0) \geq p(D^\alpha w_0 - D^\alpha v_0) - q(w_0 - v_0),
\]
\[
\text{Where } t \in (0, 1] \setminus \{t_1, \ldots, t_m\}, v_0, w_0 \in PC_{\alpha-}[0, 1] \text{ are lower and upper solutions of problem (1) – (4)};
\]
\[
(H_2) \text{ There exist constants } p, q > 0 \text{ with } p^2 > 4q \text{ such that}
\]
\[
f(t, x_2, y_2) - f(t, x_1, y_1) \geq -p(y_2 - y_1) - q(x_2 - x_1),
\]
\[
\text{Where,}
\]
\[
t \in (0, 1] \setminus \{t_1, \ldots, t_m\}, v_0(t) \leq x_1 \leq x_2 \leq w_0(t), M_1(t) \leq y_1 \leq M_2(t), i = 1, 2;
\]
\[
(H_3) \text{ If } j \in \mathbb{C}(R, R), l_j(y) \geq l_j(x), \text{ and } l_j(y) \geq l_j(x),
\]
\[
\forall v_0(t_j) \leq x \leq y \leq w_0(t_j)), j = 1, 2, \ldots, m.
\]
\[
A. \text{ Example}
\]
\[
\text{suppose that } (H_1) \text{ and } (H_3) \text{ hold. Then}
\]
\[
D^\alpha(w_0(t) - v_0(t)) - \lambda_2 (w_0(t) - v_0(t)) \geq 0, \quad t \in (0, 1].
\]
\[
B. \text{ Proof}
\]
\[
\text{Let}
\]
\[
y(t) = D^\alpha(w_0(t) - v_0(t)) - \lambda_2 (w_0(t) - v_0(t)), t \in (0, 1].
\]
\[
\text{Then by } (H_1) \text{ and } (H_3) \text{ we have}
\]
\[
D^\alpha(y(t) - \lambda_1 y(t)) = D^{2\alpha}(w_0(t) - v_0(t)) + pD^\alpha(w_0(t) - v_0(t)) + q(w_0(t) - v_0(t))
\]
\[
\geq f(t, v_0, D^\alpha v_0) - f(t, w_0, D^\alpha w_0) + pD^\alpha(w_0(t) - v_0(t)) - q(w_0(t) - v_0(t)) \geq 0
\]
\[
\lim_{t \to 0^+} t^{1-\alpha} y(t) - y(1) = \lim_{t \to 0^+} t^{1-\alpha} (\mathcal{D}^\alpha w_0(t) - \mathcal{D}^\alpha v_0(t)) - \mathcal{D}^\alpha (w_0(1) - v_0(1)) \\
- \lambda_2 \lim_{t \to 0^+} t^{1-\alpha} (w_0(t) - v_0(t)) + \lambda_2 (w_0(1) - v_0(1)) \geq 0,
\]
and
\[
\lim_{t \to t^+} (t - t_j)^{1-\alpha} (y(t) - y(t_j)) \geq \int_{t_j} \left( w_0(t_j) - \lambda_2 \left( \int_{t_j} (w_0(t_j) - v_0(t_j)) \right) \right)
\geq 0, j = 1, 2, \ldots, m.
\]

Have \( y(t) \geq 0 \) for \( t \in (0, 1] \).

Hence the proof.

3. Conclusion

This paper presented existence of solutions for nonlinear impulsive fractional differential equation with periodic boundary conditions.

References