Sum Divisor Cordial Graph

R. Elambarithi\textsuperscript{1}, R. Anitha\textsuperscript{2}
\textsuperscript{1}M.Phil. Scholar, Dept. of Mathematics, Ponnaiyah Ramajayam Inst. of Science and Tech., Thanjavur, India
\textsuperscript{2}Assistant Professor, Dept. of Mathematics, Ponnaiyah Ramajayam Inst. of Science and Tech., Thanjavur, India

Abstract: In this paper, we investigate about a sum divisor cordial graph, we prove that path, comb, jewel, crown, are sum divisor cordial graphs.

Keywords: Divisor, cordial graph, path.

1. Introduction

The divisor cordial labeling is a variant of cordial labeling. It is very interesting to investigate graph or graph families which are divisor cordial as all the graphs do not admit divisor cordial labeling \cite{1}. Here, we proved that path, jewel graph and crown graph are divisor cordial graphs.

2. Mathematical Formulation

Definition: 1.1
Let $G = (V(G), E(G))$ be a simple graph and $f: V(G) \rightarrow \{1, 2, \cdots, |V(G)|\}$ be a bijection. For each edge $uv$, assign the label 1 if $2|(f(u) + f(v))$ and the label 0 otherwise. The function $f$ is called a sum divisor cordial labeling if $|e_f(0) - e_f(1)| \leq 1$. A graph that admits a total divisor cordial labeling is named a total divisor cordial graph.

Definition: 1.2
The comb $P_n \cdot K_2$ is the graph obtained from a path by attaching a pendant edge to each vertex of the path.

Example: 1.3
The join of two graphs $G_1$ and $G_2$ is denoted by $G_1 + G_2$ and whose vertex set is $V(G_1 + G_2) = V(G_1) \cup V(G_2)$ and edge set is $E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{uv; u \in V(G_1), v \in V(G_2)\}$

Example: 1.4
The jewel $J_n$ is the graph with vertex set $V(J_n) = \{u, v, x, y, u_1: 1 \leq i \leq n\}$ and edge set $E(J_n) = \{ux, uy, xy, xv, yv, uu_1, vv_1: 1 \leq i \leq n\}$.

Example: 1.5
For a simple connected graph $G$ the square of graph $G$ is denoted by $G^2$ and defined as the graph with the same vertex set as of $G$ and two vertices are adjacent in $G^2$ if they are at a distance 1 or 2 apart in $G$.

Theorem 1.6
The path $P_n$ is sum divisor cordial graph.

Proof: Let $P_n$ be a path with consecutive vertices $v_1, v_2, \cdots, v_n$. Then $P_n$ is of order $n$ and size $n - 1$.
Define $f: V(P_n) \rightarrow \{1, 2, \cdots, n\}$ as follows:

Case (i): $n$ is odd
\[ f(v_i) = \begin{cases} i & \text{if } i \equiv 0, 1 \pmod{4} \\ i + 1 & \text{if } i \equiv 2 \pmod{4} \end{cases} \text{ for } 1 \leq i \leq n \\ i - 1 & \text{if } i \equiv 3 \pmod{4} \]

Case (ii): $n$ is even
\[ f(v_i) = \begin{cases} i & \text{if } i \equiv 1, 2 \pmod{4} \\ i + 1 & \text{if } i \equiv 3 \pmod{4} \end{cases} \text{ for } 1 \leq i \leq n \\ i - 1 & \text{if } i \equiv 0 \pmod{4} \]

In both cases the induced edge labels are $f^*(v_iv_{i+1}) = \begin{cases} 1 & \text{if } 2 | (f(v_i) + f(v_{i+1})) \\ 0 & \text{otherwise} \end{cases}$

We observe that,
\[ e_f(0) = \begin{cases} n-1 & \text{if } n \text{ is odd} \\ n & \text{if } n \text{ is even} \end{cases} \]
\[ e_f(1) = \begin{cases} n-1 & \text{if } n \text{ is odd} \\ n & \text{if } n \text{ is even} \end{cases} \]

Thus, $|e_f(0) - e_f(1)| \leq 1$.

Hence, the path $P_n$ is sum divisor cordial graph.

Theorem 1.8
The comb is sum divisor cordial graph.

Proof: Let $G$ be a comb obtained from the path $v_1, v_2, \cdots, v_n$ by joining a vertex $u_i$ to $v_i$ for each $i = 1, 2, \cdots, n$. Then $G$ is of order $2n$ and size $2n - 1$.
Define $f: V(G) \rightarrow \{1, 2, \cdots, 2n\}$ as follows:
\[ f(v_i) = 2i - 1; 1 \leq i \leq n \]
\[ f(u_i) = 2i; 1 \leq i \leq n \]

Then, the induced edge labels are $f^*(v_iv_{i+1}) = \begin{cases} 1 & \text{if } 1 \leq i \leq n - 1 \\ 0 & \text{if } 1 \leq i \leq n \end{cases}$
\[ f^*(v_iu_i) = 0; 1 \leq i \leq n \]

We observe that, $e_f(0) = n$ and $e_f(1) = n - 1$.
Thus, $|e_f(0) - e_f(1)| \leq 1$.
Hence, the comb is sum divisor cordial graph.
The jewel $J_n$ is sum divisor cordial graph.

Proof:

Let $G = J_n$. Let $V(G) = \{u, v, x, y, u_i : 1 \leq i \leq n\}$ and $E(G) = \{ux, uy, xy, xv, uv, u_iu_i : 1 \leq i \leq n\}$. Then $G$ is of order $n + 4$ and size $2n + 5$.

Define $f : V(G) \rightarrow \{1, 2, \ldots, n + 4\}$ as follows:

$$
\begin{align*}
 f(u) &= 1; \\
 f(v) &= 2; \\
 f(x) &= 3; \\
 f(y) &= 4; \\
 f(u_i) &= i + 4; \quad 1 \leq i \leq n.
\end{align*}
$$

Then, the induced edge labels are

$$
\begin{align*}
 f^*(ux) &= 1; \\
 f^*(uy) &= 0; \\
 f^*(xy) &= 0; \\
 f^*(vx) &= 0; \\
 f^*(vy) &= 1; \\
 f^*(u_iu_{2i-1}) &= 1; \quad 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \\
 f^*(u_2v_{2i}) &= 0; \quad 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \\
 f^*(vuv_{2i-1}) &= 0; \quad 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \\
 f^*(u_2v_{2i}) &= 1; \quad 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor
\end{align*}
$$

We observe that, $e_f(0) = n + 3$ and $e_f(1) = n + 2$.

Thus, $|e_f(0) - e_f(1)| \leq 1$.

Hence, $J_n$ is sum divisor cordial graph.

Theorem 1.10

The gear $G_n$ is sum divisor cordial graph.

Proof:

Let $G = G_n$. Let $V(G) = \{v, u_i, v_i : 1 \leq i \leq n\}$ and $E(G) = \{vv_i, v_iu_i : 1 \leq i \leq n; uiv_{i+1} : 1 \leq i \leq n - 1; u_nv_1\}$. Then $G$ is of order $2n + 1$ and size $3n$.

Define $f : V(G) \rightarrow \{1, 2, \ldots, 2n + 1\}$ as follows:

$$
\begin{align*}
 f(v) &= 1; \\
 f(v_{2i-1}) &= 4i - 1; \quad 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \\
 f(v_{2i}) &= 4i; \quad 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \\
 f(u_{2i-1}) &= 4i - 2; \quad 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \\
 f(u_{2i}) &= 4i + 1; \quad 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor
\end{align*}
$$

Then, the induced edge labels are

$$
\begin{align*}
 f^*(vv_{2i-1}) &= 1; \quad 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \\
 f^*(vv_{2i}) &= 0; \quad 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \\
 f^*(v_iu_i) &= 0; \quad 1 \leq i \leq n \\
 f^*(u_iv_{i+1}) &= 1; \quad 1 \leq i \leq n - 1 \quad \text{if $n$ is odd} \\
 f^*(u_nv_1) &= 1 \quad \text{if $n$ is even}
\end{align*}
$$

We observe that, $e_f(0) = \left\lfloor \frac{3n}{2} \right\rfloor$ and $e_f(1) = \left\lfloor \frac{3n}{2} \right\rfloor$

Thus, $|e_f(0) - e_f(1)| \leq 1$.

Hence, $G_n$ is sum divisor cordial graph.

3. Conclusion

The divisor cordial labeling is a variant of cordial labeling. Here, we discuss about the sum divisor cordial graph, and also prove that path, comb, jewel, crown, are sum divisor cordial graphs.

References