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The Zero Divisor Graph of a Lattice

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***Abstract*: The properties and diameter and girth of the zero divisor graph Γ(L) where L is a Lattice, discuss finding dominating set and the domination number of the zero divisor graph Γ(L).**

***Keywords*: Divisor, lattice, bipartite**

1. Introduction

The development of many branched in mathematics has been necessitated while considering certain real life problems arising in practical life on problems arising in other science.

The concept of zero-divisor graph of a commutative ring was introduces by Beck [1], but this work was mostly concerned with colourings of rings.

These are many papers which interlink graph theory and lattice theory. These papers discuss the properties of graphs derived from partially ordered sets and lattices. S. K. Nimbhokar, M. P. Wasadikar and M. M. Pawar have introduced the notion of coloring in graphs derived from lattices. In, Estaji and K. Khashyarmanesh associated to any finite lattice L, a simple graph G(L) whose vertex set is 〖Z(L)^* and two vertices P and Q are adjacent if and only if P∧ Q = 0. They are connections between the zero divisor graphs of lattices and rings and obtained some basic properties of the zero divisor graph of a lattice.

2. Mathematical Formulation

Definition 2.1

The lattice with the least element 0 be represent as $(L,\vee,\wedge)$. Then,$(a \in L)$ is called an atom. When$(x \in L)$ is 0. such that $0 < x < a$. Here, the set of all atoms of L is denoted by $A(L)$. The lattice L be atomic if for any $x \in L$, there exists an element $a \in A(L)$ such that $a \leq y$.

Definition 2.2

An element $x \in L$ is irreducible if $y = a \vee b$ implies $y = a$ or $y = b$ for $a, b \in L$.

Definition 2.3

P is said to satisfy the ascending chain condition(ACC), if given any sequence $x_1 \leq x_2 \leq \cdots \leq x_n \leq \cdots$ of elements of P there exists $k \in N$ such that $x_k = x_{k+1} = \cdots$.

Definition 2.4

The dual of the arranging chain condition is the descending chain condition (DCC)

Definition 2.5

$p \in L$, the annihilator of p is defined by $Ann(p) = \{q \in L: p \wedge \mathrm{q}\}$.

Definition 2.6

A nonempty subset I of L is called an ideal if $p, q \in I$ implies $p \vee q \in I$ and $\ell \in L$, $p \in I$ and $\ell \leq p$ imply $\ell \in I$.

Definition 2.7

A proper ideal I of L is said to be prime if $p, q \in L$ and $p \wedge q \in I$ imply$p \in I$ or $q \in I$.

Theorem 2.8

Let L be a Boolean algebra. Then L is finite if and only if its set of atoms is finite.

Theorem 2.9

Every finite Boolean algebra is atomic.

Theorem 2.10

Let L be an atomic Boolean algebra. Then 1 is the lub of the set of all atoms.

Proposition 2.11

If L is a lattice, then $diam(\Gamma(\mathrm{L})) \leq 3$.

Theorem 2.12

Let L be a distributive lattice with 0 and $\varphi(\Gamma(\mathrm{L})) < \infty$. Then L has only a finite number of distinct minimal prime ideals, $P_i, 1 \leq i \leq\ m$. It satisfy $\bigcap_{i=1}^{m} P_i = 0$ and $\bigcap_{i \neq j} P_i \neq 0$ for all j. Further, no element of $L - \bigcup_{i=1}^{m} P_i$ is a zero divisor and every minimal prime ideal of L has the form $Ann(p)$ for some$p \in L$.

Lemma 2.13

If a distributive complemented lattice contains an infinite increasing chain, then $(\Gamma(\mathrm{L})) < \infty$.

Lemma 2.13

If for some $p, q \in L$, $Ann(x)$ and $Ann(y)$ are distinct prime ideals then $p \wedge q = 0$.

Proposition 2.14

Let L be a lattice satisfying DCC.

(i) If $p, q \in L^*$ and $p \wedge q = 0$, then there exists a non zero $z \in J(L)$ such that $z \in N(x)$ and $z \notin N(y)$.
(ii) If $\Gamma(L)$ is complete, then $Z(L) \subseteq J(L)$.

Proof:

Let $x, y \in L^*$ and $x \wedge y = 0$. Let $S = \{z \in Z(L)^*: z \leq y \text{ and } z \wedge x = 0\}$. Clearly S is non empty as $y \in S$. Since L satisfies DCC, there exists a minimal element $z \in S$. Suppose $z = y_1 \vee y_2$ with $y_1 < z$ and $y_2 < z$. Then $y_1 \wedge x = 0 = y_2 \wedge x$ simply $y_1, y_2 \in S$, a contradiction to the minimality of z. Hence z is join-irreducible and z satisfies the required property in $\Gamma(L)$.

(i) Assume that $\Gamma(L)$ is complete. For $x, y \in Z(L)^*$, the set S defined in the proof of (i) is a singleton set for every $x \in Z(L)^*$. Again by the proof of (i) $Z(L) \subseteq J(L)$.

Theorem 2.15

Let L be a distributive lattice. Then $\Gamma(L)$ is a complete bipartite graph if and only if there exist prime ideals P_1 and P_2 in L such that $P_1 \cap P_2 = \{0\}$.

Proof:

Assume that $\Gamma(L)$ is a complete bipartite graph with bipartition (V_1, V_2). Set $P_1 = V_1 \cup \{0\}$ and $P_2 = V_2 \cup \{0\}$. Clearly $P_1 \cap P_2 = \{0\}$. Let $x_1, x_2 \in P_1$. If $x_1 = 0$ or $x_2 = 0$, then $x_1 \vee x_2 \in P_1$. Let $x_1, x_2 \neq 0$. Since $\Gamma(L)$ is complete bipartite, for any $y \in V_2$, $x_1 \wedge y = 0$ and $x_2 \wedge y = 0$ and so $(x_1 \vee x_2) \wedge y = 0$. Then $x_1 \vee x_2 \in P_1$. If $z \in L - \{0\}$, $x \in P_1$ and $z \leq x$, then for any $y_1 \in P_2$, $z \wedge y_1 = 0$ and so $z \in P_1$. Thus P_1 is an ideal. Let $x_1 \wedge x_2 \in P_1$. Then for any $y \in V_2$, $x_1 \wedge x_2 \wedge y = 0$. If $x_2 \wedge y = 0$, then $x_2 \in P_1$. If $x_2 \wedge y \neq 0$, since $x_2 \wedge y \in V_2$, we conclude that $x_1 \in P_1$. Thus P_1 is prime. Similarly P_2 is a prime ideal.

Conversely assume that there exist prime ideals P_1 and P_2 such that $P_1 \cap P_2 = \{0\}$. Set $V_1 = P_1 - \{0\}$ and $V_2 = P_2 - \{0\}$. For $x, y \in Z(L)^*$, let $x \wedge y = 0$. Since P_1 and P_2 are prime ideals, without loss of generality $x \in P_1$ and $y \in P_2$. Then $x \in V_1$ and $y \in V_2$. Clearly $x \wedge y \in P_1 \cap P_2$ and so $x \wedge y = 0$. Hence x and y are adjacent in $\Gamma(L)$. From this we get that $Z(L)^* = V_1 \cup V_2$ and $\Gamma(L)$ is a complete bipartite graph. Consider the lattice L given below. It contains two prime ideals $P_1 = \{0, a, b\}$, $P_2 = \{0, c, d\}$ and $P_1 \cap P_2 = \{0\}$. Note that the zero divisor graph $\Gamma(L)$ is a complete bipartite graph.

3. Conclusion

We present some results on these types of graphs and also we discuss about the zero divisor graph of a lattice and properties of the zero divisor graph of a lattice, domination in Γ(L), diameter and girth of Γ(L).

References

[1] I. Beck, Coloring of Commutative Rings, J. Algebra, 116, (1988), 208 - 226.