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Algebraic Properties of S-Normal and Polynomial Matrices

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***Abstract*: The concept of polynomial and s-normal matrices is used to obtain some equivalent condition on s-Normal and polynomial matrices. Further, Numerical examples are proved.**

***Keywords*: s-normal, polynomal matrices.**

1. Introduction

The concept of normal matrices plays an important role in the spectral theory of matrices and in the theory of generalized inverses. polynomial matrices have significant importance in quantum mechanics because they preserve norm. Recently, Krishnamoorthy and Vijayakumar [1] obtained equivalent conditions on s-normal matrices. Also, some properties of secondary unitary matrices have been studied by Krishnamoorthy and Govindarasu [2].

2. Mathematical Formulation

Definition: 1.1

For $A = [a_{ij}] \in C^{n\times n}$, the secondary transpose of A is denoted by A^s and is defined as

$A^S = [a_{n-j+1,n-i+1}]$ i , j = 1 to n.

$$\text{Let } A = \begin{bmatrix} 2+3i & 2+4i & 3i-4 \\ 8+4i & 2-3i & 2i \\ 4i-1 & 6+3i & 6-2i \end{bmatrix}$$

$$A^s = \begin{bmatrix} 6-2i & 2i & 3i-4 \\ 6+3i & 2-3i & 2+4i \\ 4i-1 & 8+4i & 2+3i \end{bmatrix}$$

Definition: 1.2

Any square matrix is said to be permutation matrix if its secondary diagonal elements all are 1 and other elements are 0.

Remark: 1.3

If V is a permutation matrix then,

$V^T = V = V^* = V$ & $V^2 = I_n$.

$VV^T = V^TV = I_n$

Example: 1.4

$$V = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \text{ is a permutation matrix of order 3.}$$

Remark: 1.5

For any $A \in C^{n\times n}$, $A^S = VA^TV$

Example: 1.6

$$A = \begin{bmatrix} 2-4\,i & 4 \\ 7 & 5+3i \end{bmatrix}$$

$$VA^TV = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2-4\,i & 4 \\ 7 & 5+3i \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 5+3\,i & 4 \\ 7 & 2-4i \end{bmatrix}$$

Definition: 1.7

A matrix $A \in C^{n\times n}$ is said to be secondary symmetric, if $A = A^s$.

Example: 1.8

$$A = \begin{bmatrix} 2+\,i & 1 \\ 2 & 2-i \end{bmatrix}$$

$$A^s = \begin{bmatrix} 2+\,i & 1 \\ 2 & 2-i \end{bmatrix}$$

$$A^\theta = \begin{bmatrix} 2+\,i & 1 \\ 2 & 2-i \end{bmatrix} \text{ and}$$

$A = A^\theta$

Thus A is s-hermitian.

Remark: 1.9

For, any $A \in C^{n\times n}$,

(i) $(A^\theta)^\theta = A$.

(ii) $(A+B)^\theta = A^\theta + B^\theta$.

(iii) $(AB)^\theta = B^\theta A^\theta$.

Definition: 1.10

Any matrix $A \in C^{n\times n}$ is said to be similar to a matrix $B \in C^{n\times n}$, if there exists a non singular matrix P such that

$A = P^{-1}BP$.

Definition: 1.11

A polynomial matrix is a matrix whose elements are Polynomial.

Example: 1.12

$$\text{Let } A(x) \begin{bmatrix} 2+x^2 & x+x^2 \\ 1+x & 3+x+x^2 \end{bmatrix}$$

$= A_2 x^2 + A_{1x} + A_0$ be a polynomial matrix.

$A(A^{-1})^\theta (A^{-1} A\theta) = I_n$

$A(A^\theta)^{-1} (A^{-1} A^\theta) = I_n$

$A^\theta A = AA^\theta$

A is s-normal.

Hence it is proved.

Hence the theorem

Definition: 2.1

A matrix $A \in C^{n\times n}$ is said to be secondary normal (s-normal) matrix if

$AA^{\theta} = A^{\theta}A.$

$A = \begin{bmatrix} 1+3i & 0 \\ 0 & 2+4i \end{bmatrix}$

$A^{s} = \begin{bmatrix} 2+4i & 1 \\ 2 & 1+3i \end{bmatrix}$

$A^{\theta} = A = \begin{bmatrix} 2-4i & 1 \\ 2 & 1-3i \end{bmatrix}$

$A A^{\theta} = \begin{bmatrix} 1+3i & 0 \\ 0 & 2+4i \end{bmatrix} \begin{bmatrix} 2-4i & 0 \\ 0 & 1-3i \end{bmatrix}$

$= \begin{bmatrix} 14+2i & 0 \\ 0 & 14-2i \end{bmatrix}$

$= A A^{\theta}$

Thus A is s-normal matrix.

Definition: 2.3

A matrix $A \in C^{n\times n}$ is said to be s-unitary matrix if

$A A^{\theta} = A^{\theta}A = I_n.$

Remark: 2.4

A matrix $A \in C^{n\times n}$ is s-unitary

$\rightarrow A^{-1} = A^{\theta}$ or $A^{-1} = VA^{*}V$

$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ is an S- unitary matrix.

$A^{\theta} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$

$AA^{\theta} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

Thus A is an S-Unitary matrix.

In the following we shall see the definition of secondary unitarily (s-unitarily) equivalent matrices.

Definition: 2.5

Let $A,B \in C^{n\times n}$. Then the matrix B is said to be s-unitarily equivalent to the matrix A , if there exists an s-unitary matrix U such that

$B = UAU^{\theta}$

Example: 2.6

Also, $U^{\theta}U = I_2$

Therefore, $UU^{\theta} = U^{\theta}U = I_2$ and U is s-unitary matrix.

Let $A = \begin{bmatrix} 1+i & 2i \\ 3+2i & 3 \end{bmatrix}$

$B = \begin{bmatrix} 2+2i & 2+3i \\ -2+2i & -3+2i \end{bmatrix}$

For, $U = \begin{bmatrix} \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{bmatrix}$

$UU^{\theta} = \begin{bmatrix} \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{bmatrix}$

Also $UU^{\theta} = I_2$

Now $U^{\theta}UA = \begin{bmatrix} -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1+i & 2i \\ 3+2i & 3 \end{bmatrix} \begin{bmatrix} \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{bmatrix}$

$= \begin{bmatrix} 2+2i & 2+3i \\ -2+2i & -3+2i \end{bmatrix}$

Hence B is s-unitarily equivalent to A.

Theorem: 2.7

If $A \in C^{n\times n}$ is s-unitarily equivalent to a diagonal matrix D, then A is s-normal matrix.

Proof:

Since A is s-unitarily equivalent to a diagonal matrix D, then there exists a s-unitary matrix P, such that

$A = PDP^{\theta}.$

Now,

$AA^{\theta} = (PDP^{\theta})(PDP^{\theta})^{\theta}$

$= PD(P^{\theta} P)D^{\theta}P^{\theta}$

$= P(DD^{\theta})P^{\theta}$

$= P(D^{\theta}D)P^{\theta}$

$= (PD^{\theta})I_n(DP^{\theta})$

$= (PD^{\theta}P^{\theta})(PDP^{\theta})$

$= (PDP^{\theta})^{\theta}(PDP^{\theta})$

$= A^{\theta}A$

Thus $AA^{\theta} = A^{\theta}A$ and A is s-normal.

Theorem: 2.8

A matrix $A \in C^{n\times n}$ is s-normal if and only if $A^{-1} A^{\theta}$ is s-unitary matrix.

Proof:

$A^{-1} A^{\theta}$ is s-unitary

$(A^{-1} A^{\theta}) (A^{-1} A^{\theta})^{\theta} = I_n$

$(A^{-1} A^{\theta}) (A(A^{-1})^{\theta}) = I_n$

$A^{-1}(A^{\theta}A) (A^{\theta})^{-1} = I_n$

$A^{\theta}A = AA^{\theta}$

A is s–normal.

Similarly, $A^{-1} A^{\theta}$ is s-unitary

$(A^{-1} A^{\theta})^{\theta} (A^{-1} A^{\theta}) = I_n$

$A(A^{-1})^{\theta}$ (A-1 Aθ) $= I_n$

$A(A^{\theta})^{-1} (A^{-1} A^{\theta}) = I_n$

$A^{\theta}A = AA^{\theta}$

A is s-normal. Hence it is proved.

3. Conclusion

In matrix theory, the special types of matrices namely s-normal and polynomial matrices are taken. We generalized some of the results are proved on s-normal and polynomial matrices.

References

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