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Theoretical Measures on Hilbert Spaces

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***Abstract*: In this paper, we investigate the theoretical measures on Hilbert spaces and its related theorems are proved. This 3D space was applied on geophysical vibration signals from the process of drilling of rock.**

***Keywords*: Hilbert Spaces, inner product space, orthonormal.**

1. Introduction

The highest degree of generalization and abstraction of the physical space represent the classes of functional spaces called Hilbert spaces. The definition of Hilbert space is the following: Hilbert space is a complete space with inner product. It shows infinite-dimensional and complex. Hilbert Spaces are Inner Product Spaces, with rich geometric structures because of the orthogonality of vectors in the space. Hilbert space is a structure which combines the familiar ideas from vector spaces with the right ideas of analysis to form a useful context for handling mathematical problems of a wide variety [1].

2. Mathematical formulation

Theorem: 2.1

If v is a Borel measure on **T** such that

$$\int_T^\infty \chi_n dv = 0 \quad for\ n > 0,$$

then v is absolutely continuous and there exists f in H^1 such that $dv = fd\varphi$

Proof:

If μ denotes the total variation of v, then exist a Borel function ψ

Such that

$dv = \psi d\mu$ and $|\psi| = 1 a.e$ with respect to μ

If M denotes the closed subspace of $L^2(\mu)$

Spanned by $\{\chi_m : m > 0\}$ then

$$(\chi_m, \bar{\psi}) = \int_T^0 \chi_m \psi d\mu = \int_T^0 \chi_m dv = 0$$

And hence $\bar{\psi}$ is orthogonal to M in $L^2(\mu)$

Suppose M =$M_1 \oplus M_2$ is the decomposition by

Let E is the Borel subset of **T** given by

$M_1 = L_E^2(\mu)$

$$\mu(E) = \int_T^0 |\psi|^2 I_E d\mu = (\bar{\psi}, \bar{\psi} I_E) = 0$$

Since $\psi \overline{I_E}$is in $M_1$and $\bar{\psi}$ is orthogonal to M Therefore M_1={0}and there exist a μ measurable function φ such that

$M = \varphi H^2 |\varphi|^2 d\mu = {d\varphi}/{2\pi}$

since χ_1 is in M, it follows that there exists

g in H^2 such that $\chi_1 = \varphi g$ with respect to μ and since $\varphi \neq 0 \mu\ a.e.$

then we have μ is mutually absolutely continuous with lebesgue measure.

If function f is in L^1(**T**) such that $dv = fd\varphi$

Then by hypothesis f in H^1

Hence the proof.

Note:

M_∞ is the maximal ideal space in the commutative Banach algebra H^∞ let D be a open unit disk in M_∞ For z in D define the bounded linear functional $\varphi_z$in H^1 such that

$\varphi = (f) = \frac{1}{2\pi}\int_0^{2\pi} \frac{f(e^{i\varphi})}{1-ze^{-i\varphi}} d\varphi$ for f in H^1

Since the function $1/(1 - ze^{-i\varphi})$ is in $L^\infty(\boldsymbol{T})$ and H^1 is contained in $L^1(\boldsymbol{T})$ it follows that φ_z is a bounded linear functional on H^1.

Moreover since $1/(1 - ze^{-i\varphi}) = \sum_{m=0}^{\varphi} e^{-im\varphi} z^m$ and the latter series converges absolutely,

$$\varphi_z(f) = \sum_{k=0}^{\infty} z^k (\frac{1}{2\pi}\int_0^{2\pi} f\chi_k d\varphi)$$

Thus if p is analytic trigonometric polynomial then $\varphi_z(P) = P(z)$ and φ_z is a multiplicative linear functional on P_+.

Lemma:

If f and g are in H^2 and z is inn D then fg is in H^1 and $\varphi_z(fg) = \varphi_z(f)\varphi_z(g)$

Proof:

Let $\{p_m\}_{m=1}^\infty$ and $\{q_m\}_{m=1}^\infty$ be sequences of analytic trigonometric polynomials such that $\lim_{m\to\infty} ||f - p_m||_2 = ||g - q_m||_2 = 0$

Since the product of two L^2 functions is in L^1, we have

$$||fg - p_m q_m||_1 \leq ||fg - p_m g||_1 + ||p_m g - p_m q_m||_1$$
$$\leq ||f - p_m||_2 ||g||_2 + ||p_m||_2 ||g - q_m||_2$$

and hence $\lim_{m\to\infty} ||fg - p_m q_m||_1 = 0$. Since each $p_m q_m$ is in H^1 we have f_g in H^1

Since φ_z is continuous, we have

$$\varphi_z(fg) - \lim_{m\to\infty} \varphi_z(p_m q_m) = \lim_{m\to\infty} \varphi_z(p_m) \lim_{m\to\infty} \varphi_z(q_m)$$
$$= \varphi_z(f)\varphi_z(g)$$

Hence the proof.

Theorem: 2.2

For z in D the restriction of φ_z to H^∞ is a multiplicative linear functional on H^∞. Moreover mapping F from D in to H^∞ defined by $F(z) = \varphi$ is homeomorphism.

Proof:

That φ_z restricted to H^∞ is a multiplicative linear functional follows from the proceeding lemma.

Fixed f in H^1, the function $\varphi_z(f)$ is analytic in z, it follows that F is continuous. Moreover since

$\varphi_z(\chi_1) = z$ it follows that f is one to one.

Finally, if $\{\varphi_{z\propto}\}_{\propto\in A}$ is a net in M_∞

Converging to φ_z, then

$$\lim_{\propto\in A} z_\propto = \lim_{\propto\in A} \varphi_{z\propto}(\chi_1) = \varphi_z(\chi_1) = z$$

Hence F is a homeomorphism.

Note:

1. f is a gelfand transform of H^∞ of function f_0

For f in H^1 we define f on D by

$$f(z) = \varphi_z(f)$$

Theorem: 2.3

A function f in H^2 is an outer function if clos[fP_+] =H^2.

Proposition:

A function φ in $H^\infty$is invertible in H^∞ if and only if φ is invertible in L^∞ and is an outer function.

Proof:

If $1/\varphi$ is in H^∞, then obviously φ is invertible in L^∞.

Moreover since $clos[\varphi P_+] = \varphi H^2 \supset \varphi(\frac{1}{\varphi} H^2) = H^2$

It follows that φ is an outer function.

Conversely, if $1/\varphi$ is in $L^\infty(T)$ and φ is an outer function, then

φH^2=clos[φ P_+]= H^2. Therefore there exists a function ψ in H^2 such that $\varphi\psi = 1$ and hence $1/\varphi$ is in H^∞ hence φ is invertible.

Hence the proof.

Theorem: 2.4

If f is a function in $L^2(T)$ then there exists an outer function g such that $|f| = |g|$a.e. if and only if clos[fP_+] is simply invariant subspace for $M_{\chi 1}$.

Proof:

If $|f| = |g|$ for some outer function g; then $f = \varphi g$ for some unimodular function φ in $L^\infty(T)$ then

clos[fP_+] = clos[φgP_+] = clos[φgP_+] = φH^2

and hence clos[fP_+] is simply invariant.

Conversely, if clos[fP_+] is simply invariant for $M_{\chi 1}$ then there exists a unimodular function φ in $L^\infty(T)$ Such that clos[fP_+]=φH^2

Since f is in clos[fP_+]

There must exist function φ in H^2

Such that $f = \varphi g$

Hence g or f, g is outer function.

Corollary:

If f is a function in $L^2(T)$ such that $|f| \geq \varepsilon > 0$ then there exist an outer function go such $|g| = |f|$

Proof:

Let M = clos[fP_+] then $M_{\chi 1}M$ is the closure of $\{f_p : p \in P_+, p(0) = 0\}$

If we calculate the distance from f to such an f_p we find that

$||f - fp||_2^2 = \frac{1}{2\pi}\int_0^{2\pi}|f|^2|1-p|^2 d\varphi \geq \frac{\epsilon^2}{2\pi}\int_0^{2\pi}|1-p|^2 d\varphi \geq \varepsilon^2$.

And hence f is not in $M_{\chi 1}M$

Therefore, M is simply invariant and hence there exist an outer function g such that $|g| = |f|$.

3. Conclusion

The study of Hilbert spaces and its related theorems are discussed. Hilbert spaces provide special geometrical properties are discussed and it is useful for drilling of rocks.

References

[1] L. Mate, *Hilbert space methods in science and engineering*, Adam Hilger, 1989.