One Dimensional Heat Equations and Two Dimensional Steady Heat Flow Equations and their Applications

Anthony Muthondu Kinyanjui[1], Francis Muli[2], Joseph Njuguna Karomo[3]

[1,3]Department of Pure and Applied Sciences, Kirinyaga University, Kerugoya, Kenya
[2]Department of Mathematics and Computer Science, Catholic University of East Africa, Nairobi, Kenya

Abstract: **The heat equation is an important partial differential equation which describes the distribution of heat (or variation in temperature) in a given region over time. This study, therefore, seeks to give a deeper insight into the 1-D heat equations and their applicability. Application of Boundary Element Method in a solution of potential heat problems (Two-dimensional heat equation) is also explored. The study used the Boundary Element Method (BEM) of numerical approximation and compared the results to the analytical solution. Nodes taken within the solution domain were compared to the analytical solutions. The study confirmed that the numerical values agreed fairly well with the analytical values. The temperature and flux profiles were also found to be close to the analytical profile.**

Keywords: **Boundary Element Method (BEM), Boundary Integral Equation (BIE), Finite Element Method (FEM), Finite Volume Method (FVM), Temperature fluxes, Boundary Value Problems, Initial conditions.**

1. Introduction

In this research, we have made idealized assumptions, such as constant thermal properties of our perfectly circular geometry, to facilitate a tractable solution. We first consider heat transfer in 1-D and then the potential equation. The study also assumed that the steady-state temperature has been reached so that any time dependency is ignored; this also implies that the heat flux is constant at the particular region of interest. Despite all these simplifications, the solution can still give a good insight into how heat may be diffused throughout the region. This study used the BEM method of numerical approximation and compared the results to the analytical solution. FORTRAN 95 software was used to compute results and graph using G-Sharp [1].

A. Governing Equations: Derivation in One Dimension

The heat equation is derived from Fourier's law and conservation of energy [2]. By Fourier's law, the flow rate of heat energy through a surface is proportional to the negative temperature gradient across the surface,

$$q = -k\nabla u \tag{1}$$

Where k is the thermal conductivity and u is the temperature. In one dimension, the gradient is an ordinary spatial derivative,

and so Fourier's law is,

$$q = -ku_x \tag{2}$$

In the absence of work done, a change in internal energy per unit volume in material ΔQ is proportional to the change in temperature. That is,

$$\Delta Q = c_p \rho \Delta u \tag{3}$$

Where c_p is the specific heat capacity and ρ is the mass density of the material. Choosing zero energy at temperature zero, this can be rewritten as,

$$Q = c_p \rho u \tag{4}$$

The increase in internal energy in a small spatial region of the material.

$$x - \Delta x \leq \xi \geq x + \Delta x$$

Over the time period,

$$t - \Delta t \leq \tau \geq t + \Delta t$$

Is given by

$$c_p \rho \int_{x-\Delta x}^{x+\Delta x} \{u(\xi, t + \Delta t) - u(\xi, t - \Delta t)\} d\xi =$$
$$c_p \rho \int_{t-\Delta t}^{t+\Delta t} \int_{x-\Delta x}^{x+\Delta x} \frac{\partial u}{\partial \tau} d\xi d\tau \tag{5}$$

Where the fundamental theorem of calculus was used [3]. With no work done, and absent of any heat sources or sinks, this change in internal energy in the interval $[x - \Delta x, x + \Delta x]$ is accounted for entirely by the flux of heat across the boundaries. By Fourier's law, this

$$k \int_{t-\Delta t}^{t+\Delta t} \left\{\frac{\partial u}{\partial x}(x + \Delta x, \tau) - \frac{\partial u}{\partial x}(x - \Delta x, \tau)\right\} d\tau =$$
$$k \int_{t-\Delta t}^{t+\Delta t} \int_{x-\Delta x}^{x+\Delta x} \frac{\partial^2 u}{\partial \xi^2} d\xi d\tau \tag{6}$$

Again by the fundamental theorem of calculus. By conservation of energy,

$$\int_{t-\Delta t}^{t+\Delta t} \int_{x-\Delta x}^{x+\Delta x} \{c_p \rho u_\tau - k u_{\xi\xi}\} d\xi d\tau = 0 \tag{7}$$

This is true for any rectangle $[t - \Delta t, t + \Delta t]$. Consequently, the integrand must vanish identically;

$$c_p \rho u_\tau - k u_{xx} = 0 \tag{8}$$

Or, $u_t = \frac{k}{c_p \rho} u_{xx}$

This is the one-dimensional heat equation.

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B. Internal Heat Generation

The function U above represents the temperature of a body in 1-D. Alternatively, it is sometimes convenient to change units and represent u as the heat density of a medium. Since heat density is proportional to temperature in a homogeneous medium, the heat equation is still obeyed in the new units. Suppose that a body obeys the heat equation and, in addition, generates is own heat per unit volume (e.g., in watts/L) at a rate given by a known function q varying in space and time. Then the heat per unit volume, U satisfies an equation.

$$\frac{\partial u}{\partial t} = \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) + q \qquad (9)$$

C. The Boundary Conditions

The heat equation is a second-order partial differential equation in the spatial coordinates. We need boundary conditions in order to specify how our system interacts with the outside surroundings. There are three general types of boundary conditions: Dirichlet, Neumann and Mixed boundary conditions.

1) Dirichlet Boundary Conditions

Dirichlet boundary conditions the temperature is specified at the boundary as a function of time

$$T(X = 0, t) = T_{bc1}(t) \qquad (10)$$

If the temperature is constant, then we have.

$$T(X = 0, t) = T_{bc1} \qquad (11)$$

In this case, we have a physical situation where our system is touching an infinite heat reservoir that maintains a constant temperature. In a one dimensional system, we must have two boundary conditions, one at the left-hand-side boundary and other at the right-hand-side boundary. If our system is of length L in the x-direction, then our second Dirichlet boundary condition would be of the form:

$$T(X = L, t) = T_{bc2}(t) \qquad (12)$$

2) Neumann Boundary Conditions

In Neumann boundary conditions, the heat flux is set at the boundary. These conditions are expressed as follows (for the one-dimensional case) [4].

$$\frac{dT}{dx}(X = 0, t) = \frac{dT(t)}{dx}\big|_{bc1} \qquad (13)$$

That is, at the left-hand-side boundary the one-dimensional system, the heat flux is a specified function of time. If the heat flux is constant we have:

$$\frac{dT}{dx}(X = 0, t) = \frac{dT}{dx}\big|_{bc1} \qquad (14)$$

In this case, we have the physical situation where the system is touching an infinite heat source that maintains a constant flux of heat into the system regardless of the temperature. One end of the rod is well insulated. No heat leaves it. The flux is Zero. In this case, we would use a Neumann boundary condition [5].

In a one-dimensional system, we must have two boundary conditions one at the left-hand-side boundary and the other at the right-hand- side boundary.

If our one-dimensional system is of length L in the x-direction, then our second Neumann boundary condition would be of the form:

$$\frac{dT}{dx}(X = L, t) = \frac{dT(t)}{dx}\big|_{bc2} \qquad (15)$$

3) Mixed Boundary Conditions

Mixed Boundary Conditions is a mixture of the Dirichlet and Neumann Boundary conditions. They take the following form (for the one-dimensional case) [4].

$$\frac{dT}{dx}(x = 0, t) + T(x = 0, t) = T_{bc1}(t) + \frac{dT(t)}{dx}\big|_{bc1} = f(t) \qquad (16)$$

There are very relevant physical systems which require these elaborate boundary conditions.

D. Initial Conditions

1) Generalized Initial Condition

The heat equation is first order in time zero. In general, this initial condition can be written as;

$$T(x, y, t = 0) = T_{ic}(x, y) \qquad (17)$$

2) Steady-State Initial Condition

If the temperature profile is initially a steady-state (linear) profile between two boundary condition temperature T_{bc1} and T_{bc2}, then we would have the formula for the linear interpolation between them, which in one-dimension as:

$$T(x, y, t = 0) = T_{bc1} + \frac{X}{L}(T_{bc2} - T_{bc1}) \qquad (18)$$

2. Solutions using Fourier series

The following solution technique for the heat equation was proposed by Joseph Fourier in his treatise th´eorie analytique de la chaleur [6]. Let us consider the following heat equation for one space variable. This could be used to model heat conduction in a rod.

$$u_t = ku_{xx} \qquad (19)$$

Assuming the initial condition,

$$u(0, x) = f(x) \forall x \in [0, L] \qquad (20)$$

Where the function *f* is given and the boundary conditions,

$$u(t, 0) = 0 = u(t, L) \forall t > 0 \qquad (21)$$

The solution to equation (2.1) above is:

$$u(x, t) = \sum_{n=1}^{+\infty} D_n \left(sin \frac{n\pi x}{L} \right) e^{-\frac{n^2\pi^2 kt}{L^2}} \qquad (22)$$

Where,

$$D_n = \frac{2}{L} \int_0^L f(x) sin \frac{n\pi x}{L} dx \qquad (23)$$

A. Generalizing the Solution Technique

The solution technique used above can be greatly extended to many other types of equations. The idea is that the operator U_{xx} with the zero boundary conditions can be presented in terms of its eigenvectors. This leads naturally to one of the basic ideas of the spectral theory of linear self-adjoint operators. Consider the linear operator $\Delta U = U_{xx}$. The finite sequence of function,

$$e_n(x) = \sqrt{\frac{2}{L}} sin \frac{n\pi x}{L} \text{ for } n \geq 1 \text{ are eigenvectors of } \Delta \text{ indeed,}$$

$$\Delta e_n = -\frac{n^2\pi^2}{L^2} e_n$$

Moreover, any eigenvector *f* of Δ with the boundary conditions $f(0) = f(L) = 0$ is of the form e_n for some $n \geq 1$. The functions e_n for $n \geq 1$ form an orthonormal sequence

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with respect to a certain inner product on the space of real-valued functions on $[0, L]$. This means,

$$\langle e_n, e_m \rangle = \int_0^L e_n(x)e_m(x)dx = \begin{cases} 0, & n \neq m \\ 1, & m = n \end{cases} \quad (24)$$

Finally, the sequence $\{e_n\}$, $n \in N$ spans a dense linear subspace of $L_2(0, L)$.

This shows that in effect, we have diagonalized the operator Δ.

B. Fundamental Solutions

A fundamental solution, also called a heat kernel, is a solution of the heat equation corresponding to the initial conditions of an initial point source of heat at a known position. These can be used to find a general solution of the heat equation over certain domains; In one variable, the Green's function is a solution of the initial value problem.

$$\begin{cases} u_t(x,t) - ku_{xx}(x,t) = 0, & -\infty < x < \infty, \ 0 < t < \infty \\ u(x, t = 0) = \delta(x) \end{cases}$$

$$(25)$$

Where δ is the Dirac delta function. The solution to this problem is the fundamental solution,

$$\phi(x,t) = \frac{1}{\sqrt{4\pi kt}} \exp(-\frac{x^2}{4kt}) \quad (26)$$

One can obtain the general solution of the one variable heat equation with initial condition $U(x,0) = g(x)$ for $-\infty < x < \infty$ and $0 < t < \infty$ by applying a convolution:

$$u(x,t) = \int \phi(x - y, t)g(y)dy \quad (27)$$

In several spatial variables, the fundamental solution solves the analogous problem,

$$\begin{cases} u_t(x,t) - k\sum_{i=1}^n u_{x_ix_i}(X,t) = 0 \\ u(X, t = 0) = \delta(X) \end{cases} \quad (28)$$

in $-\infty < x < \infty$ and $0 < t < \infty$. The n-variable fundamental solution is the product of the fundamental solutions in each variable; i.e.,

$$\phi(X,t) = \Phi(x_1, t)\Phi(x_2, t)\ldots\ldots\ldots\Phi(x_n, t) = \frac{1}{(4\pi kt)^{\frac{n}{2}}} e^{-\frac{X.X}{4kt}} \quad (29)$$

The general solution of the heat equation on $\Re^n$ is then obtained by a convolution so that to solve the initial value problem with $u(x, 0) = g(x)$, one has,

$$u(X,t) = \int_{\Re^n} \phi(X - y, t)g(y)d^n y \quad (30)$$

The general problem on a domain Ω in $\Re^n$ is

$$\begin{cases} u_t(X,t) - k\sum_{i=1}^n u_{x_ix_i}(X,t) = 0, \ X \in \Omega & 0 < t < \infty \\ u(X, t = 0) = g(X), & X \in \Omega \end{cases} \quad (31)$$

With either Dirichlet or Neumann boundary data. A Green's function always exists, but unless the domain Ω can be readily decomposed into one variable problem, it may not be possible to write it down explicitly. The method of the image provides one additional technique for obtaining Green's function solutions in 1- D.

A variety of elementary Green's function solutions in one-dimension are recorded here. In some of these, the spatial domain is the entire real line $(-\infty, \infty)$. In others, it is the semi-finite interval $(0, \infty)$ with either Neumann or Dirichlet boundary conditions. One further variation is that some of these

solve the inhomogeneous equation.

$$U_t = kU_{xx} + f \quad (32)$$

Where f is some given function of x and t.
Initial value problem on $(-\infty, \infty)$

$$\begin{cases} u_t = ku_{xx}, & -\infty < x < \infty, \ 0 < t < \infty \\ u(x, 0) = g(x), & \text{IC:} \end{cases} \quad (33)$$

$$U(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \exp(-\frac{(x-y)^2}{4kt})g(y)dy \quad (34)$$

Initial value problem on $(0, \infty)$ with homogeneous Dirichlet boundary conditions.

$$\begin{cases} u_t = ku_{xx}, & 0 \leq x < \infty, \ 0 < t < \infty \\ u(x, 0) = g(x), & \text{IC;} \\ u(0, t) = 0 & \text{BC.} \end{cases} \quad (35)$$

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_0^{\infty} \left\{ \text{esp}\left(-\frac{(x-y)^2}{4kt}\right) - \exp\left(-\frac{(x+y)^2}{4kt}\right)\right\} g(y)dy \quad (36)$$

Initial value problem on $(0, \infty)$ with homogeneous Neumann boundary conditions.

$$\begin{cases} u_t = ku_{xx}, & 0 \leq x < \infty, \ 0 < t < \infty \\ u(x, 0) = g(x), & \text{IC;} \\ u_x(0, t) = 0 & \text{BC.} \end{cases} \quad (37)$$

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_0^{\infty} \left\{ \text{esp}\left(-\frac{(x-y)^2}{4kt}\right) + \exp\left(-\frac{(x+y)^2}{4kt}\right)\right\} g(y)dy \quad (38)$$

Problem on $(0, \infty)$ with homogeneous initial conditions and non-homogeneous Dirichlet boundary condition.

$$\begin{cases} u_t = ku_{xx}, & 0 \leq x < \infty, \ 0 < t < \infty \\ u(x, 0) = g(x), & \text{IC;} \\ u(0, t) = h(t) & \text{BC.} \end{cases} \quad (39)$$

$$u(x,t) = \int_0^t \frac{x}{\sqrt{4\pi k(t-s)^3}} \exp(-\frac{x^2}{4k(t-s)})h(s)d(s) \quad (40)$$

Inhomogeneous heat equation problem on $(-\infty, \infty)$ homogeneous initial conditions.

$$\begin{cases} u_t = ku_{xx} + f(x,t), & -\infty < x < \infty, 0 < t < \infty \\ u(x, 0) = 0, & \text{IC} \end{cases} \quad (41)$$

$$U(x,t) = \int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi k(t-s)}} \exp\left(-\frac{(x-y)^2}{4k(t-s)}\right) f(y, s)dyds \quad (42)$$

Problem on $(0, \infty)$ with homogeneous Dirichlet boundary conditions and initial conditions.

$$\begin{cases} u_t = ku_{xx} + f(x,t), & -\infty < x < \infty, 0 < t < \infty \\ u(x, 0) = 0, & \text{IC;} \\ u(0, t) = 0, & \text{BC} \end{cases} \quad (43)$$

$$U(x,t) = \int_0^t \int_0^{\infty} \frac{1}{\sqrt{4\pi k(t-s)}} \left\{ \exp\left(-\frac{(x-y)^2}{4k(t-s)}\right) - \exp\left(-\frac{(x+y)^2}{4k(t-s)}\right)\right\} f(y, s)dyds \quad (44)$$

3. Two Dimensional Heat Equation

We consider a steady-state two-dimensional heat equation for simplicity. In this case, we use the potential functions $(\nabla^2 U = 0)$ which can be derived from Gauss' Theorem, also

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referred to as divergence theorem.

The partial differential operator, ∇^2 or Δ is called the Laplace operator, or just Laplacian [7].

The starting point for our model is the Fourier's Law (19) which specifies that heat transfer is governed by the equation 1.

If the system is in steady-state, then Conservation of Energy (19) implies that, in the absence of heat sinks or sources, the heat flux throughout the region must satisfy:

$$\nabla \cdot q = 0 \qquad (45)$$

In one dimension, this would imply that heat flux must be constant at all points; in more than one dimension, it implies that heat flux entering a control region must equal heat flux leaving the same region. Without any heat generation and considering 2-D heat flow in steady-state, the conservation of energy equation (9) reduces down to Laplace's equation.

$$\nabla^2 U = 0 \qquad (46)$$

or

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0 \qquad (47)$$

Since U is frequently a potential function, this equation is also known as a potential equation.

A boundary value problem which is of practical interest requires solving the Laplace equation (3.3) in the two-dimensional region R bounded by a simply closed curve Γ subject to the boundary conditions

$$U = f_1(x,y) \text{ for } (x,y) \in \Gamma_1$$
$$\frac{\partial U}{\partial n} = f_2(x,y) \text{ for } (x,y) \in \Gamma_2 \qquad (48)$$

Where f_1 and f_2 are suitably prescribed functions and Γ_1 and Γ_2 are non-intersecting curves such that $\Gamma_1 U \Gamma_2 = \Gamma$

It is worth noting that the Laplace equation is both linear and homogeneous while the boundary conditions we are specifying are linear and non-homogeneous. This creates a problem because the separation of variables as discussed in 1-D heat equation requires homogeneous boundary conditions. The boundary conditions in the equation. (48) imply that at each and every point on the boundary, either the temperature or the heat flux (but not both) is known. To determine the temperature field in the region R, one has to solve the equation (47) to find the solution that satisfies the prescribed boundary condition on Γ.

The normal derivative $\frac{\partial U}{\partial n}$ is defined by

$$\frac{\partial U}{\partial n} = n_x \frac{\partial U}{\partial x} + n_y \frac{\partial U}{\partial y} \qquad (49)$$

Here the unit normal vector $[n_x, n_y]$ on Γ is taken to be pointing away from the region R. It is worth noting that the vector may vary from point to point on Γ. Thus, $[n_x, n_y]$ is a function of x and y. The boundary conditions prescribed in equation.(48) are assumed to be well-posed so that the boundary value problem has a unique solution, that is, it is assumed that one can always find a function $U(x,y)$ satisfying equations (47) and (48) and that there is only one such function.

In general, it is difficult (if not impossible) to solve exactly the BVP defined by equations (3.3) and (3.4). The mathematical complexity depends on the geometrical shape of the region R and the boundary conditions given in (3.4). Exact solutions can only be found for relatively simple geometries or R (such as square and circular regions) together with particular boundary conditions. For more complicated geometries, one may have to resort to numerical (approximate) techniques.

A. The Transformation from Cartesian Coordinates to Polar Coordinates

We consider equation (47) above and the parametric equations:

$x = r\cos\theta$ and $y = r\sin\theta$; where r is the radius of circular geometry and thus a constant.

Partially differentiating U w.r.t x gives

$$\frac{\partial U}{\partial x} = \frac{\partial U}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial U}{\partial \theta}\frac{\partial \theta}{\partial x}$$

$$\frac{\partial^2 U}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial U}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial U}{\partial \theta}\frac{\partial \theta}{\partial x}\right) = \frac{\partial^2 U}{\partial r\partial x}\frac{\partial r}{\partial x} + \frac{\partial U}{\partial r}\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 U}{\partial \theta\partial x}\frac{\partial \theta}{\partial x} + \frac{\partial U}{\partial \theta}\frac{\partial^2 \theta}{\partial x^2} \qquad (50)$$

Applying the chain rule:

$$\frac{\partial^2 U}{\partial r\partial x} = \frac{\partial^2 U}{\partial r^2}\frac{\partial r}{\partial x} + \frac{\partial^2 U}{\partial \theta\partial r}\frac{\partial \theta}{\partial x} \qquad (51)$$

$$\frac{\partial^2 U}{\partial \theta\partial x} = \frac{\partial}{\partial \theta}\left(\frac{\partial U}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial U}{\partial \theta}\frac{\partial \theta}{\partial x}\right) = \frac{\partial^2 U}{\partial r\partial \theta}\frac{\partial r}{\partial x} + \frac{\partial^2 U}{\partial \theta^2}\frac{\partial \theta}{\partial x} \qquad (52)$$

Next, we look for $\frac{\partial r}{\partial x}$ and $\frac{\partial \theta}{\partial x}$

$r = \sqrt{(x^2+y^2)}$ and $\theta = \tan^{-1}\frac{y}{x}$ hence

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{r} \qquad (53)$$

$$\frac{\partial \theta}{\partial x} = \frac{\partial}{\partial x}\left(\tan^{-1}\frac{y}{x}\right) = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right) = -\frac{y}{r^2}$$

$$\frac{\partial^2 r}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial \theta}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{x}{r}\right) = \frac{1}{r} - \frac{x}{r^2}\left(\frac{x}{r}\right) = \frac{r^2-x^2}{r^3} = \frac{y^2}{r^3} \qquad (54)$$

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial \theta}{\partial x}\right) = \frac{\partial}{\partial x}\left(-\frac{y}{r^2}\right) = \frac{\partial}{\partial r}\left(-\frac{y}{r^2}\right)\frac{\partial r}{\partial x} = \frac{2xy}{r^4} \qquad (55)$$

Substituting equations (51) to (55) into equation (50) and simplifying gives;

$$\frac{\partial^2 U}{\partial x^2} = \frac{x^2}{r^2}\left(\frac{\partial^2 U}{\partial r^2}\right) - \frac{2xy}{r^3}\frac{\partial^2 U}{\partial r\partial \theta} + \frac{y^2}{r^3}\frac{\partial U}{\partial r} + \frac{2xy}{r^4}\frac{\partial U}{\partial \theta} + \frac{y^2}{r^4}\frac{\partial^2 U}{\partial \theta^2} \qquad (56)$$

Similarly;

$$\frac{\partial^2 U}{\partial y^2} = \frac{y^2}{r^2}\left(\frac{\partial^2 U}{\partial r^2}\right) + \frac{2xy}{r^3}\frac{\partial^2 U}{\partial r\partial \theta} + \frac{x^2}{r^3}\frac{\partial U}{\partial r} - \frac{2xy}{r^4}\frac{\partial U}{\partial \theta} + \frac{x^2}{r^4}\frac{\partial^2 U}{\partial \theta^2} \qquad (57)$$

Add equations (56) and 57 gives;

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = \frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2}$$

We, therefore, have the Laplace equation (47) in polar form as

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} = 0 \qquad (58)$$

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Under the above assumptions, the equation (47) is independent of θ, hence it becomes;

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} = 0 \qquad (59)$$

Whose solution is: $U(r) = A\ lnr + B$

Where A and B can be determined if we know $U(r)$ and $\frac{\partial U}{\partial r}$ on the boundary of the circle. This solution blows up as $r \to \infty$ so it is only of limited interest in estimating the temperature distribution near the pipe. When the boundary at ground level is introduced, it becomes clear we cannot parameterize the solution by r alone. The purpose of highlighting this solution is that it provides the basis for the Fundamental Solution of Laplace's Equation at point $p = (x_p, y_p)$ which is given as.

$$G(x, y | x_p, y_p) = -\frac{1}{4\pi}\left[(x_p - x)^2 + (y_p - y)^2\right] \qquad (60)$$

G is defined everywhere in $\mathbb{R}^2$ apart from at (x_p, y_p) where it is singular. We can now transform the PDE equation (3.15) into a Boundary Integral Equation (BIE) which we can solve numerically on the boundary of our circular geometry.

An application of the divergence theorem to convert the double integral over R into a line integral over Γ gives

$$\int_{\Gamma}\left[\left(U_2\frac{\partial U_1}{\partial x} - U_1\frac{\partial U_2}{\partial x}\right)n_x + \left(U_2\frac{\partial U_1}{\partial y} - U_1\frac{\partial U_2}{\partial y}\right)n_y\right]ds(x, y) = 0 \qquad (61)$$

B. Numerical Methods

There are three broad classes of methods that are commonly used for boundary value problems across many fields of physics, mathematics and engineering – the finite difference method (FDM), the finite element method (FEM) and the boundary element method (BEM). The choice of BEM is informed by the fact that it can be applied where any potential problem is governed by a differential equation that satisfies the Laplace equation. Given the advantages of this approach, we shall now use this method to obtain a solution for equation (59). There are a couple of different ways this can be done via the Direct Method (which solves for U and $\frac{\partial U}{\partial N}$ directly) or the Indirect Method (which solves for a density function which then generates solutions for U).

C. Direct Boundary Element Method

To derive the boundary integral equation we let $U_1 = G(x, y; x_p, y_p)$ the fundamental solution defined in equation (60) and $U_2 = U$, where U is the required solution of the boundary value problem defined by equation (47) and (48).

Since $G(x, y; x_p, y_p)$ is not well defined at the point (x_p, y_p), the reciprocal relation in equation (61) is valid for $U_1 = G(x, y; x_p, y_p)$ and $U_2 = U$ only if (x_p, y_p) does not lie in the region $R \cup \Gamma$. Thus

$$\int_{\Gamma}\left[U(x, y)\frac{\partial}{\partial n}\left(G(x, y; x_p, y_p)\right) - G(x, y; x_p, y_p)\frac{\partial}{\partial n}(U(x.y))\right]ds(x, y) = 0$$

$$for\ (x_p, y_p) \notin R \cup \Gamma \qquad (62)$$

For the case in which (x_p, y_p) lies in the region R, equation (60) is valid if we replace Γ by $\Gamma \cup \Gamma_\varepsilon$, where Γ_ε is a circle of center (x_p, y_p) and radius ε. This is because $G(x, y; x_p, y_p)$ and its first-order partial derivatives (with respect to x or y) are well defined in the region between Γ and Γ_ε. Thus we can write

$$\int_{\Gamma \cup \Gamma_\varepsilon}\left[U(x, y)\frac{\partial}{\partial n}\left(G(x, y; x_p, y_p)\right) - G(x, y; x_p, y_p)\frac{\partial}{\partial n}(U(x.y))\right]ds(x, y) = 0$$

This implies that

$$\int_{\Gamma}\left[U(x, y)\frac{\partial}{\partial n}\left(G(x, y; x_p, y_p)\right) - G(x, y; x_p, y_p)\frac{\partial}{\partial n}(U(x.y))\right]ds(x, y) =$$
$$-\int_{\Gamma_\varepsilon}\left[U(x, y)\frac{\partial}{\partial n}\left(G(x, y; x_p, y_p)\right) - G(x, y; x_p, y_p)\frac{\partial}{\partial n}(U(x.y))\right]ds(x, y) \qquad (63)$$

Equation (63) holds for any radius $\varepsilon > 0$, so long as the circle Γ_ε lies completely inside the region bounded by Γ. We can let $\varepsilon \to 0^+$ in the equation (63). this gives

$$\int_{\Gamma}\left[U(x, y)\frac{\partial}{\partial n}\left(G(x, y; x_p, y_p)\right) - G(x, y; x_p, y_p)\frac{\partial}{\partial n}(U(x.y))\right]ds(x, y) =$$
$$-\lim_{\varepsilon \to 0^+}\int_{\Gamma_\varepsilon}\left[U(x, y)\frac{\partial}{\partial n}\left(G(x, y; x_p, y_p)\right) - G(x, y; x_p, y_p)\frac{\partial}{\partial n}(U(x.y))\right]ds(s, y) \qquad (64)$$

The Taylor series of $U(x, y)$ about the point (x_p, y_p) is given by

$$U(x, y) = \sum_{m=0}^{\infty}\sum_{k=0}^{m}\left(\frac{\partial^m U}{\partial x^k \partial y^{m-k}}(U(x, y))\right)\Bigg|_{(x,y)=(x_p,y_p)} \frac{(x-x_p)^k(y-y_p)^{m-k}}{k!(m-k)!}$$

On the circle Γ_ε, $r = \varepsilon$. Thus

$$U(x, y) = \sum_{m=0}^{\infty}\sum_{k=0}^{m}\left(\frac{\partial^m U}{\partial x^k \partial y^{m-k}}(U(x, y))\right)\Bigg|_{(x,y)=(x_p,y_p)} \frac{\varepsilon^m cos^k\theta sin^{m-k}\theta}{k!(m-k)!}$$

for $(x, y) \in \Gamma_\varepsilon \qquad (65)$

Similarly, we have

$$\frac{\partial}{\partial n}[U(x, y)]$$
$$= \sum_{m=0}^{\infty}\sum_{k=0}^{m}\left(\frac{\partial^m U}{\partial x^k \partial y^{m-k}}\left\{\frac{\partial}{\partial n}[U(x, y)]\right\}\right)\Bigg|_{(x,y)=(x_p,y_p)} \frac{\varepsilon^m cos^k\theta sin^{m-k}\theta}{k!(m-k)!}$$

for $(x, y) \in \Gamma_\varepsilon \qquad (66)$

Using equations (65) and (66) and writing $ds(x, y) = \varepsilon d\theta$ with θ ranging from 0 to 2π, we attempt to evaluate the limits on the right-hand side of equation (64).

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On Γ_ε, the normal vector $[n_x, n_y]$ is given by $[-cos\theta, -sin\theta]$. We also utilize trigonometric identity $cos^2\theta + sin^2\theta = 1$ to get.

$$\int_{\Gamma_\varepsilon} U(x,y)\frac{\partial}{\partial n}[G(x,y;x_p,y_p)]ds(x,y) =$$
$$-\frac{1}{2\pi}U(x_p,y_p)\int_0^{2\pi}d\theta$$
$$-\frac{1}{2\pi}\sum_{m=1}^\infty\sum_{k=0}^m\frac{\varepsilon^m}{k!(m-k)!}\left(\frac{\partial^m U}{\partial x^k\partial y^{m-k}}\right)\Big|_{(x,y)=(x_p,y_p)}$$
$$\int_0^{2\pi}cos^k\theta sin^{m-k}\theta d\theta$$
$$\rightarrow -U(x_p,y_p)\text{as }\varepsilon\rightarrow 0^+ \quad (67)$$
and

$$\int_{\Gamma_\varepsilon} G(x,y;x_p,y_p)\frac{\partial}{\partial n}[U(x.y)]ds(x,y) =$$
$$\frac{1}{2\pi}\sum_{m=1}^\infty\sum_{k=0}^m\left(\frac{\partial^m U}{\partial x^k\partial y^{m-k}}\left\{\frac{\partial}{\partial n}[U(x,y)]\right\}\right)\Big|_{(x,y)=(x_p,y_p)}$$
$$\times\frac{\varepsilon^{m+1}ln(\varepsilon)}{k!(m-k)!}\int_0^{2\pi}cos^k\theta\, sin^{m-k}\theta d\theta \quad (68)$$
$$\rightarrow 0\text{ as }\varepsilon\rightarrow 0^+\text{ since}\varepsilon^{m+1}\ln(\varepsilon)\rightarrow 0\text{ as }\varepsilon$$
$$\rightarrow 0^+\text{ for m} = 1,2,\dots\dots\dots$$

Consequently, as $\varepsilon\rightarrow 0^+$, equation (64) becomes

$$U(x_p,y_p) = \int_\Gamma\left[U(x,y)\frac{\partial}{\partial n}\left(G(x,y;x_p,y_p)\right) -\right.$$
$$\left. G(x,y;x_p,y_p)\frac{\partial}{\partial n}(U(x.y))\right]ds(x,y)$$
$$for\,(x_p,y_p)\in R \quad (69)$$

Together with the fundamental equation (65), equation (64) provides us with a boundary integral solution for 2-D Laplace's equation.

If both U and $\frac{\partial U}{\partial n}$ are known at all points on Γ, the line integral in equation (69) can be evaluated to calculate U at any point (x_p, y_p) in the interior of R.

From the boundary conditions (3.4), at any given point on Γ, either U or $\frac{\partial U}{\partial n}$, not both, is known. To solve the interior boundary problem, we must find the unknowns either U or $\frac{\partial U}{\partial n}$ on the outer and inner boundaries.

Rather than having an expression relating the value of U at some point inside the domain to boundary integrals, a more useful expression would be one relating the value of U at some point on the boundary to boundary integrals.

In case the point (x_p, y_p) lies on Γ, equation (64) holds if we replace the curve Γ by $D\ U\ D_\varepsilon$

If Γ_ε is the circle of center (x_p, y_p) and radius ε, then D is part of Γ that lies outside Γ_ε and D_ε is part of Γ_ε that lies inside R. Thus

$$\int_D\left[U(x,y)\frac{\partial}{\partial n}\left(G(x,y;x_p,y_p)\right) -\right.$$
$$\left. G(x,y;x_p,y_p)\frac{\partial}{\partial n}(U(x.y))\right]ds(x,y)$$

$$\Gamma_\varepsilon = -\int_{D_\varepsilon}\left[U(x,y)\frac{\partial}{\partial n}\left(G(x,y;x_p,y_p)\right)\right.$$
$$\left. - G(x,y;x_p,y_p)\frac{\partial}{\partial n}(U(x.y))\right]ds(x,y) \quad (70)$$

As $\varepsilon\rightarrow 0^+$, the curve D in equation (70) tends to Γ. Thus we write

$$\int_\Gamma\left[U(x,y)\frac{\partial}{\partial n}\left(G(x,y;x_p,y_p)\right) -\right.$$
$$\left. G(x,y;x_p,y_p)\frac{\partial}{\partial n}(U(x.y))\right]ds(x,y)$$
$$= -\lim_{\varepsilon\rightarrow 0^+}\int_{D_\varepsilon}\left[U(x,y)\frac{\partial}{\partial n}\left(G(x,y;x_p,y_p)\right) -\right.$$
$$\left. G(x,y;x_p,y_p)\frac{\partial}{\partial n}(U(x.y))\right]ds(x,y) \quad (71)$$

We expect D_ε to tend to a semi-circle as $\varepsilon\rightarrow 0^+$, if (x_p, y_p) lies on a smooth part of Γ. It then follows that in attempting to evaluate the limit on the right-hand side of equation (70), we have to integrate over only half a circle.

Equations (67) and (68) becomes

$$\lim_{\varepsilon\rightarrow 0^+}\int_{D_\varepsilon} U(x,y)\frac{\partial}{\partial n}[G(x,y;x_p,y_p)]ds(x,y) =$$
$$-\frac{1}{2}U(x_p,y_p),$$
$$\lim_{\varepsilon\rightarrow 0^+}\int_{D_\varepsilon} G(x,y;x_p,y_p)\frac{\partial}{\partial n}[U(x.y)]ds(x,y) = 0$$

Hence equation (71)becomes

$$\frac{1}{2}U(x_p,y_p) = \int_\Gamma\left[U(x,y)\frac{\partial}{\partial n}\left(G(x,y;x_p,y_p)\right) -\right.$$
$$\left. G(x,y;x_p,y_p)\frac{\partial}{\partial n}(U(x.y))\right]ds(x,y) \quad (72)$$
for(x_p, y_p) lying on a smooth part of Γ

Together with the boundary conditions in equation (48), equation (72)can be utilized to obtain a numerical procedure for determining the unknown U and or $\frac{\partial U}{\partial n}$ on the boundary Γ. Once U and $\frac{\partial U}{\partial n}$ are known at all points on Γ, the solution of the interior boundary value problem defined equations (47) and (48) is given by equation (69) at any point (x_p, y_p) inside R.

For convenience, we may write equations (62),(69)and (72) as a single equation given as

$$\beta(x_p,y_p)U(x_p,y_p) = \int_\Gamma\left[U(x,y)\frac{\partial}{\partial n}\left(G(x,y;x_p,y_p)\right) -\right.$$
$$\left. G(x,y;x_p,y_p)\frac{\partial}{\partial n}(U(x.y))\right]ds(x,y) \quad (73)$$

D. Numerical Discretization of The BIE

We transform the integral equation (3.29) into a system of algebraic equations. The boundary Γ is discretized into N small straight line elements $\Gamma^{(1)}, \Gamma^{(2)}, \dots\dots, \Gamma^{(N-1)}$ and $\Gamma^{(N)}$
That is $\Gamma \simeq \Gamma^{(1)} U \Gamma^{(2)} U \dots\dots\dots. U \Gamma^{(N-1)} U \Gamma^{(N)}$ [8] (74)

These straight lines are called boundary elements. The center of each element is referred to as a collocation node.

To construct the boundary elements, we put N-well spaced out points $(x^{(1)}, y^{(1)}), (x^{(2)}, y^{(2)}), \dots\dots\dots, (x^{(N-1)}, y^{(N-1)})$and $(x^{(N)}, y^{(N)})$on Γ, in the order given following counter-clockwise direction.

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Defining $\left(x^{(N+1)}, y^{(N+1)}\right) = \left(x^{(1)}, y^{(1)}\right)$, we take $\Gamma^{(k)}$ to be the boundary element from $\left(x^{(k)}, y^{(k)}\right)$ to $\left(x^{(k+1)}, y^{(k+1)}\right)$ for $k = 1,2, \dots \dots \dots, N$

For an approximation of U and $\frac{\partial U}{\partial n}$ on the boundary Γ, we assume that these functions are constants over each of the boundary elements. Specifically, we make the following approximation:

$$U \simeq \overline{U}^{(k)} \text{ and } \frac{\partial U}{\partial n} = \overline{P}^{(K)} \text{ for}(x, y) \in \Gamma^{(k)}(k = 1,2, \dots \dots \dots, N) \qquad (75)$$

Where $\overline{U}^{(k)}$ and $\overline{P}^{(K)}$ are respectively the values of U and $\frac{\partial U}{\partial n}$ at the mid-point of $\Gamma^{(k)}$.

With equations (74) and (75), equation (73) can be written as

$$\beta\left(x_p, y_p\right)U\left(x_p, y_p\right) = \sum_{k=1}^{N}\left\{\overline{U}^{(k)}F_2^{(k)}\left(x_p, y_p\right) - \overline{P}^{(K)}F_1^{(k)}\left(x_p, y_p\right)\right\} \qquad (76)$$

where

$$F_1^{(k)}\left(x_p, y_p\right) = \int_{\Gamma^{(k)}} G(x, y; x_p, y_p)ds(x, y)$$

$$F_2^{(k)}\left(x_p, y_p\right) = \int_{\Gamma^{(k)}} \frac{\partial}{\partial n}\left[G\left(x, y; x_p, y_p\right)\right]ds(x, y) \qquad (77)$$

For a given k, either $\overline{U}^{(k)}$ or $\overline{P}^{(K)}$ is known from the boundary conditions in equation (48). Thus, there are N unknown constants on the right-hand side of equation (78). To determine their values, we have to generate N equations containing the unknowns.

We let (x_p, y_p) in equation (3.32) be given in turn by the midpoints of $\Gamma^{(1)}, \Gamma^{(2)}, \dots \dots, \Gamma^{(N-1)}$ and $\Gamma^{(N)}$ and obtain;

$$\frac{1}{2}\overline{U}^{(m)} = \sum_{k=1}^{N}\left\{\overline{U}^{(k)}F_2^{(k)}\left(\overline{x}^{(m)}, \overline{y}^{(m)}\right) - \overline{P}^{(k)}F_1^{(k)}\left(\overline{x}^{(m)}, \overline{y}^{(m)}\right)\right\} \qquad (78)$$

for $m = 1,2, \dots \dots \dots, N$, and $\left(\overline{x}^{(m)}, \overline{y}^{(m)}\right)$ is the midpoint of $\Gamma^{(m)}$

In the derivation of equation (3.34), we take $\beta\left(\overline{x}^{(m)}, \overline{y}^{(m)}\right) = \frac{1}{2}$, since $\left(\overline{x}^{(m)}, \overline{y}^{(m)}\right)$ being the midpoint of $\Gamma^{(m)}$ lies on a smooth part of the approximate boundary i.e. $\Gamma^{(1)} U \Gamma^{(2)} U \dots \dots \dots U \Gamma^{(N-1)} U \Gamma^{(N)}$.

Equation (78) constitutes a system of N linear algebraic equations containing the N unknowns on the right-hand side of equation (78). Since both $\overline{U}^{(k)}$ and $\overline{P}^{(K)}$ contain known as well as unknown boundary data, it is necessary to write the equations with all unknowns appearing on one side as given below.

$$\sum_{k=1}^{N} a^{(mk)}Z^{(k)} = \sum_{k=1}^{N} b^{(mk)} \text{ for } m = 1,2, \dots \dots \dots, N \quad (79)$$

Where $a^{(mk)}, b^{(mk)}$ and $Z^{(k)}$ are defined by

$$a^{(mk)} = \begin{cases} -F_1^{(k)}\left(\overline{x}^{(m)}, \overline{y}^{(m)}\right) \text{ if } U \text{ is specified over } \Gamma^{(k)} \\ F_2^{(k)}\left(\overline{x}^{(m)}, \overline{y}^{(m)}\right) - \frac{1}{2}\delta^{(mk)} \text{if} \frac{\partial U}{\partial n} \text{ is specified over} \Gamma^{(k)} \end{cases}$$

$$b^{(mk)} = \begin{cases} \overline{U}^{(k)}\left(-F_2^{(k)}\left(\overline{x}^{(m)}, \overline{y}^{(m)}\right) + \frac{1}{2}\delta^{(mk)}\right) \\ \quad \text{if } U \text{ is specified over} \Gamma^{(k)} \\ \overline{P}^{(k)}F_1^{(k)}\left(\overline{x}^{(m)}, \overline{y}^{(m)}\right) \text{if} \frac{\partial U}{\partial n} \text{ is specified over} \Gamma^{(k)} \end{cases}$$

$$\delta^{(mk)} = \begin{cases} 0 \text{ if } m \neq k \\ 1 \text{ if } m = k \end{cases}$$

$$Z^{(k)} = \begin{cases} \overline{P}^{(k)} \text{if } U \text{ is specified over } \Gamma^{(k)} \\ \overline{U}^{(k)} \text{if} \frac{\partial U}{\partial n} \text{ is specified over} \Gamma^{(k)} \end{cases} \qquad (80)$$

We notice that by collocating the load point $U(x_p, y_p)$ with the nodes $k = 1, \dots \dots \dots \dots, N$, we get N system of equations.

Equation (76) with $\beta\left(x_p, y_p\right) = 1$ provides us with an explicit formula for computing U in the interior of R that is;

$$U\left(x_p, y_p\right) \simeq \sum_{k=1}^{N}\left\{\overline{U}^{(k)}F_2^{(k)}\left(x_p, y_p\right) - \overline{P}^{(K)}F_1^{(k)}\left(x_p, y_p\right)\right\} \text{ for } \left(x_p, y_p\right) \in R \qquad (81)$$

4. Methods of Solution and Discussion of Results

This section presents computation and discussion of results on heat distribution around a simply connected circular geometry. Rearranging equation (81) according to the specified conditions, we develop BEM code for the computation of the solution. Temperature and fluxes are specified on some boundaries and the values computed with the numerical model. The results are then compared to the analytical solution to ascertain the efficiency of the BEM.

Consider steady heat transfer to the exterior of circular geometry. We deal with Neumann boundary conditions.
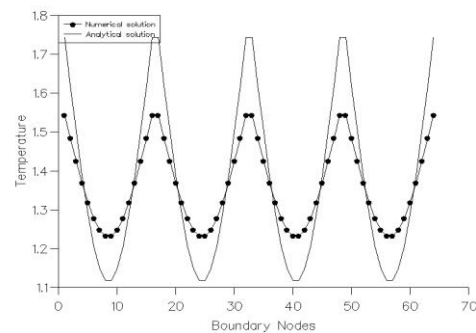

Fig. 1. Approximate and analytical temperature profiles for 64 nodes

A. Discussion of the results

The temperature distribution around a circular pipe presents a sine curve. This is in agreement with research carried out by [9]. They made the conclusion that temperature distribution around a circular pipe under isotropic condition presents a sine curve regardless of the physical conditions of the circular geometry.

In our case above, we have Cauchy boundary conditions at inner and outer boundaries. A few nodes taken within the solution domain were compared to the analytical solutions. It's evident that the numerical values agreed fairly well with the analytical values.

5. Conclusion

It is evident that many physical phenomena can be modeled using partial differential equations in particular heat transfer. In many cases, an analytical solution is not enough thus we rely on numerical solutions to obtain more information on the inherent problems. In this paper, we have observed that the application of numerical methods is limited to the cases where the functions under consideration are well behaved. We have demonstrated that BEM is a powerful numerical method that involves discretizing the boundary of a potential problem so as to get solutions that are in line with the experimental results. The temperature and flux profiles were close to the analytical profile.

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