Fractional Integro – Differential Systems in Complete Vector Space for Controllability

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***Abstract*: In this work, through the use of the theory of fractional calculus, fixed point technique and a new concept called (β,u) resolvent family, we have been established the controllability result of a class of fractional evolution nonlocal impulsive quasi-linear delay integro-differential system in a complete vector space.**

***Keywords*: Fractional intrgro-differential systems, Controllability, Nonlocal and impulsive conditions, (β,u)-resolvent family, Complete vector space, Fixed point theorem.**

1. Introduction

In the past decade, many authors investigated the existence result for fractional evolution equation; see [25, 26]. Moreover, there are different type of mild solutions that have been proved. For example, the first one was constructed in terms of a probability density function given by El-Borai [27] and was then developed by Zhou et al. [28, 29], and the second one was presented in terms of an β-resolvent family provided by Araya et al. [30] and then Mophou et al. [31]. But, in both senses, if the closed operator in the evolution equation is dependent on more then the considered case can be taken as an open problem. For this reason, we will introduce in this article a new concept called (β, u) - resolvent family, which is based on Araya-Lizama concepts [30], and Hill-phillips principles [32]. Our paper is organized as follows. Section 2 is devoted to a review of some essential results in fractional calculus and the resolvent operators that will be used in this work to obtain our main results. In section 3, we state and prove the controllability result. Section 4 deals with an example to illustrate the abstracts.

2. Preliminaries

Consider the fractional integro-differential control system of the form

$$\frac{d^{\beta}u(t)}{dt^{\beta}} + A\big(t, u(t)\big)u(t) = (B\mu)(t) + \varphi\left(t, f\left(t, u\big(\gamma(t)\big)\right), \int_0^t g\left(t, s, u\big(\delta(s)\big)\right) ds\right) \quad \ldots\ldots\ldots\ldots.. (1)$$

$$u(0)+h(u) = u_0, \quad \ldots \quad (2)$$

$$\Delta u(t_i) = I_i(u(t_i)), \quad \ldots \quad (3)$$

Where the state u(.) takes values in the Complete Vector Space X, $0 < \alpha \leq 1, t \in [0, a]$, $u_0 \in X$, i = 1,2,…m and 0 $< t_1 < t_2 < t_3 < \cdots t_m < a$. We assume that –A(t,.) is a closed linear operator defined on a dense domain D(A) in X into X such that D(A) is independent of t. It is assumed also that –A(t,.) generates an evolution operator in the Complete Vector Space X, the control function μ belongs to the space $L^2(S, U)$, a Complete Vector Space of admissible control functions with U as a Complete Vector Space and B : U→X is a bounded linear operator. The functions f : $S \times X^2 \to X$, g : $\Lambda \times X^k \to X$, φ : $S \times X^2 \to X$, $h:$ PC(S, X) $\to X$, $u(\gamma) = (u(\gamma_1), \ldots u(\gamma_r))$, $u(\delta) = (u(\delta_1), \ldots u(\delta_k))$, and $\gamma_p, \delta_q : S \to S$ are given, where p = 1,2,…,r, q = 1,2,…,k. Here S = [0, a] and $\Lambda = \{(t,s): 0 \leq s \leq t \leq a\}$.

Let $pc(S, X)$ consist of functions u from S into X, such that $u(t)$ is continuous at $t \neq t_i$ and left continuous at $t = t_i$ and the right limit $u(t_i^+)$ exists for $i = 1,2, \ldots m$. clearly $pc(S, X)$ is a Complete Vector Space with the norm $\|u\|_{pc} = sup_{t\in j}\|u(t)\|$, and let $\Delta u(t_i) = u(t_i^+) - u(t_i^-)$ constitute an impulsive condition.

In recent years, fractional differential equations have attracted the attention of many mathematician and physicists, see for instance, Baleanu et al. [1-3], Agarwal and Lakshmikantham et al. [5-8] and Kilbas et. al. [9,10]. See also [11-15]. The existence results to evolution equations with nonlocal conditions in Banach spach was studied first by Byszewski [16, 17]. Deng [18] indicated that, using the nonlocal condition $u(0) + h(u) = u_0$ to describe for instance, the diffusion phenomenon of a small amount of gas in a transparent tube can give better result than using the usual local Cauchy problem$u(0) = u_0$. Let as observe also that since Deng's papers, the function h is considered

$$h(u) = \sum_{k=1}^{p} c_k u(t_k), \quad \ldots\ldots\ldots\ldots (4)$$

Where c_k, $k = 1,2, \ldots, p$ are given constants and $0 \leq t_1 < \cdots < t_p \leq a$.

Let X and Y be two Complete Vector Spaces such that Y is densely and continuously embedded in X. For any Complete Vector Space Z, the norm of Z is denoted by$\|.\|_z$. The space of all bounded linear operators from X and Y is denoted by $B(X, Y)$ and $B(X, X)$ is written as $B(X)$.

A. Definition

The fractional integral of order $\beta > 0$ is defined by

$$I_a^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(s)}{(t-s)^{1-\beta}} ds$$

Where Γ is the gamma function and $f \in L^1([a,b],\mathbb{R}^+)$.

If $a = 0$, we can write $I^{\alpha} f(t) = (g_\beta * f)(t)$, where

$$g_\beta(t) = \begin{cases} \frac{1}{\Gamma(\beta)} t^{\beta-1}, t > 0 \\ 0\ , \qquad t \le 0 \end{cases}$$

As usual, * denotes the convolution of functions, also we have $lim_{\beta\to 0} g_\beta(t) = \delta(t)$, which is the delta function.

Definition

The Riemann-Liouville fractional derivative of order $n-1 < \beta < n$ is defined by

$${}_aD_t^{\beta} f(t) = \frac{1}{\Gamma(n-\beta)} \frac{d^n}{dt^n} \int_a^t \frac{f(s)}{(t-s)^{1+\beta-n}} ds,$$

Where f is an abstract continuous function on the interval $[a,b]$ and $n \in \mathbb{N}^*$, also the Caputo fractional derivative of order $n-1 < \beta < n$ is defined by

$${}_a^cD_t^{\beta} f(t) = \frac{1}{\Gamma(n-\beta)} \int_a^t \frac{f^{(n)}(s)}{(t-s)^{1+\beta-n}} ds.$$

This definition is still now a basic source for all authors that are working in the field of fractional calculus.

Definition

A two parameter family of bounded linear operators $U(t,s), 0 \le s \le t \le a$, on X is called an evolution system if the following two conditions are satisfied

(i)$U(t,t) = I, U(t,r)U(r,s) = U(t,s)$ for $0 \le s \le r \le t \le a$

(ii) $(t,s) \to U(t,s)$ is strongly continuous for $0 \le s \le t \le a$

Let E be the Complete Vector Space formed from D(A) with the graph norm, since $-A(t)$ is closed operator, it follows that $-A(t)$ is in the set of bounded operator from E to X.

B. Definition

Let $A(t,u)$ be a closed and linear operator with domain D(A) defined on a Complete Vector Space X and $\beta > 0$. let $\rho[A(t,u)]$ be the resolvent set of $A(t,u)$ the generator of an (β,u)-resolvent family if there exist $\omega \ge 0$ and a strongly continuous function $R_{(\beta,u)}: \mathbb{R}_+^2 \to L(X)$ such that $\{\lambda^\beta : Re(\lambda) > \omega\} \subset \rho(A)$ and for $0 \le s \le t \le \infty$,

$$\lambda^\beta I - A(s,u))^{-1} v = \int_0^\infty e^{-\lambda(t-s)} R_{(\beta,u)}(t,s) v\, dt, \ \ Re(\lambda) > \omega, (u,v) \in X^2.$$

In this case , $R_{(\beta,u)}(t,s)$ is called the (β,u)-resolvent family generated by A(t,u).

C. Definition

Let mild solution

$$\frac{d^\beta u(t)}{dt^\beta} + A(t,u(t))u(t) = (B\mu)(t) + \varphi(t, f(t,u(\gamma(t))), \int_0^t g(t,s,u(\delta(s)))ds \quad \ldots\ldots\ldots\ldots (1)$$

$$\Delta u(t_i) = I_i(u(t_i)), \quad \ldots\ldots\ldots\ldots\ldots (3)$$

(1)-(3) we mean a function $u \in PC(S{:}X)$ with values in Ω satisfying the integral equation

$$u_\mu(t) = R_{(\beta,u)}(t,0)u_0 - R_{(\beta,u)}(t,0)h(u) + \int_0^t R_{(\beta,u)}(t,s)[(B\mu)(s) + \phi(s, f(s,u(\gamma(s))), \int_0^s g(s,\eta,u(\delta(\eta)))d\eta)]ds + \sum_{0<t_i<t} R_{(\beta,u)}(t,t_i) I_i(u(t_i)), t \in J$$

For all $u_0 \in X$ and admissible control $\mu \in L^2(S,U)$. we assume the following conditions.

(H_1) The bounded linear operator $E: L^2(S,U) \to X$ defined by

$$E_\mu = \int_0^a R_{(\beta,u)}(a,s) B_\mu(s) ds,$$

Has an induced inverse operator $\tilde{E}^{-1}$ which takes values in $L^2(S,U)/\ker E$ and there exists positive constants M_1, M_2, such that $\|B\| \le M_1$ and $\|\tilde{E}^{-1}\| \le M_2$.

$(H_2) h: pc(S{:}\Omega) \to Y$ is lipschitz continuous in X and bounded in Y, that is there exists $K_1 > 0$ and $K_2 > 0$ such that

$$\|h(u)\|_\delta \le K_1,$$
$$\|h(u) - h(v)\|_\gamma \le K_2 \max_{t\in S} \|u - v\|_{PC}, \qquad u, v \in PC(S{:}X).$$

For conditions $(H_3) - (H_5)$ let Z taken as both X and Y.

$(H_3) g: \Lambda \times z^k \to z$ is continuous and there exist constants $K_3 > 0$ and $K_4 > 0$ such that

$$\int_0^t \|g(t,s,u_1,\ldots,u_k) - g(t,s,v_1,\ldots,v_k)\|_z ds \le k_3 \sum_{q=1}^k \|u_q - v_q\|_z, \quad u_q, v_q \in X, q = 1,2,\ldots,k,$$

$$k_4 = \max\{\int_0^t \|g(t,s,0,\ldots,0)\|_z ds : \ (t,s) \in \Lambda\}.$$

$(H_4) f: S \times z^r \to z$ is continuous and there exist constants $k_5 > 0$ and $k_6 > 0$ such that

$$\|f(t,u_1,\ldots,u_r) - f(t,v_1,\ldots,v_r)\|_z \le k_5 \sum_{p=1}^r \|u_p - v_p\|_z, \quad u_p, v_p \in X, p = 1,2,\ldots,r,$$

$$k_6 = \max_{t\in J} \|f(t,0,\ldots,0)\|_z.$$

$(H_5)\ \emptyset: S \times Z^2 \to Z$ is continuous and there exists constants $k_7 > 0$ and $k_8 > 0$ such that

$$\|\emptyset(t,u_1,u_2)-\emptyset(t,v_1,v_2)\|_z \leq k_7(\|u_1-v_1\|_z + \|u_2-v_2\|_z),\ u_1,u_2,v_1,v_2 \in X,$$
$$k_8 = \max_{t\in S}\|\emptyset(t,0,0)\|_z.$$

(H_6) $\gamma_p, \delta_q : S \to S$ are bijective absolutely continuous and there exist constants $c_p > 0$and $b_q > 0$ such that $\gamma_p'(t) \geq c_p$ and $\delta_p'(t) \geq b_q$ respectively for $t \in S, p = 1, \ldots r$ and $q = 1, \ldots k$.

$(H_7) I_i : X \to X$ are continuous and there exist constants $I_i > 0, i = 1,2, \ldots m$ such that $\|I_i(u) - I_i(v)\| \leq I_i\|u - v\|,\ \ u, v \in X$.

Let us take $M_0 = \max\left\|R_{(\beta,u)}(t,s)\right\|_{B(z)},\ 0 \leq s \leq t \leq a, u \in \Omega$.

(H_8) There exist positive constants $C_1, C_2, C_3 \in (0, C/3]$ and $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in [0, \frac{1}{4})$ such that

$$C_1 = M_0\|u_0\| + M_0 k_1,$$
$$C_2 = M_0 M_1 M_2[\|u_1\| + M_0\|u_0\| + M_0 k_1 + M_0 k_7\theta + M_0 k_8 a + M_0\xi]a,$$
$$C_3 = M_0 k_7\theta + M_0 k_8 a + M_0\xi,$$

And

$$\lambda_1 = ka\|u_0\| + k_1 ka + M_0 k_2,$$
$$\lambda_2 = 2a^2 k M_1 M_2\{\|u_1\|_y + M_0(\|u_0\|_y + k_1 + k_7\theta + k_8 a + \xi)\},$$
$$\lambda_3 = ka(k_7\theta + k_8 a) + M_0 k_7\rho,$$
$$\lambda_4 = ka\xi + M_0\sum_{i=1}^{m} I_i,$$

Where $\rho = a[k_5(1/c_1 + \cdots + 1/c_r) + k_3(1/b_1 + \cdots + 1/b_k)], \theta = \rho\delta + a(k_4 + k_6)$ and $\xi = \sum_{i=1}^{m}(I_i C + \|I_i(0)\|)$.

D. Definition

We shall say that the fractional system (1)-(3) is controllability on the interval S if for all $u_0, u_1 \in X$, there exists a control $\mu \in L^2(S, U)$, such that the mild solution u(.) of (1)-(3) corresponding to μ, verifies:$u(0) + h(u) = u_0, \Delta u(t_i) = I_i(u(t_i)), i = 1,2, \ldots m$ and $u_\mu(a) = u_1$.

3. Controllability result

Lemma: 3.1

Let $R_{(\beta,u)}(t,s)$ be the (β,u)-resolvant family for the fractional problem (1)-(3). There exists a constant $k > 0$such that

$$\left\|R_{(\beta,u)}(t,s)\omega - R_{(\beta,v)}(t,s)\omega\right\| \leq k\|\omega\|_Y \int_s^t \|u(\tau) - v(\tau)\|d\tau,$$

For every $u, v \in Pc(S:X)$ with values in Ω and every $\omega \in Y$.

Proof:

Since the resolvent operator is similarly to the evolution operator for nonautonomous differential equations in a Complete Vector Space, then we can use a similar manner as in [41, lemma 4.4,p.202].

Let $s_C = \{u : u \in PC(S:X), u(0) + h(u) = u_0, \Delta u(t_i) = I_i(u(t_i)), \|u\| \leq C\}$, for $t \in S, C > 0, u_0 \in X$ and $i = 1, \ldots, m$.

Theorem: 3.2

Suppose that the operator $-A(t,u)$ generates an (β,u)-resolvent family with $\left\|R_{(\beta,u)}(t,s)\right\| \leq Me^{-\sigma(t,s)}$ for some constants M, $\sigma > 0$. If hypothesis $(H_1) - (H_8)$ are satisfied, then the fractional control integro-differential system (1) with nonlocal condition (1,2) and impulsive condition (1,3) is controllable on J.

Proof

Using hypothesis (H_1), for an arbitrary function $u(.)$, we define the control

$$\mu(t) = \tilde{E}^{-1}[u_1 - R_{(\beta,u)}(a,0)u_0 + R_{(\beta,u)}(a,0)h(u) - \int_0^a R_{(\beta,u)}(a,s)\phi(s, f(s, u(\gamma(s))), \int_s^0 g(s,\eta,u(\delta(\eta)))d\eta)ds - \sum_{i=1}^{m} R_{(\beta,u)}(a,t_i)I_i(u(t_i))](t)$$

We define an operator $p : s_C \to s_C$ by

$$(Pu_\mu)(t) = R_{(\beta,u)}(t,0)u_0 - R_{(\beta,u)}(t,0)h(u) + \int_0^t R_{(\beta,u)}(t,\eta)B\tilde{E}^{-1}\Big[u_1 - R_{(\beta,u)}(a,0)u_0 + R_{(\beta,u)}(a,0)h(u) - \int_0^a R_{(\beta,u)}(a,s)\emptyset(s, f(s,u(\gamma(s))), \int_0^s g(s,\tau,u(\delta(\tau)))d\tau)ds - \sum_{i=1}^{m} R_{(\beta,u)}(a,t_i)I_i(u(t_i))\Big](\eta)d\eta + \int_0^t R_{(\beta,u)}(t,s)\emptyset(s, f(s,u(\gamma(s))), \int_0^s g(s,\tau,u(\delta(\tau)))d\tau)ds + \sum_{0<t_i<t} R_{(\beta,u)}(t,t_i)I_i(u(t_i)).$$

Using this controller we shall show that operator P has a fixed point. This fixed point is then a solution of equation.

Clearly $Pu_\mu(a) = u_1$, which means that the control μ steers system (1)-(3) from the initial state u_0 to u_1 in time a, provided e can obtain a fixed point of the nonlinear operator p.

Now we show that p maps s_C into itself.

$$\begin{aligned}&\left\|(Pu_\mu)(t)\right\|\\&\leq \|R_{(\beta,u)}(t,0)u_0\| + \|R_{(\beta,u)}(t,0)h(u)\|\\&+\int_0^t \|R_{(\beta,u)}(t,\eta)\|\|B\tilde{E}^{-1}\|\Bigg[\|u_1\|+\|R_{(\beta,u)}(a,0)u_0\|\\&+\|R_{(\beta,u)}(a,0)h(u)\|+\int_0^a\|R_{(\beta,u)}(a,s)\|\\&\times\left\{\left\|\emptyset(s,f\left(s,u(\gamma(s))\right),\int_0^s g\left(s,\tau,u(\delta(\tau))\right)d\tau)\right.\right.\\&\left.\left.-\emptyset(s,0,0)\right\|+\|\emptyset(s,0,0)\|\right\}ds\\&+\sum_{i=1}^m\|R_{(\beta,u)}(a,t_i)\|\{\|I_i(u(t_i))-I_i(0)\|+\|I_i(0)\|\}\Bigg]d\eta\\&+\int_0^t\|R_{(\beta,u)}(t,s)\|\\&\times\left\{\left\|\emptyset(s,f\left(s,u(\gamma(s))\right)\int_0^s g\left(s,\tau,u(\delta(\tau))\right)d\tau)\right.\right.\\&\left.\left.-\emptyset(s,0,0)\right\|+\|\emptyset(s,0,0)\|\right\}ds\\&+\sum_{0<t_i<t}\|R_{(\beta,u)}(t,t_i)\|\{\|I_i(u(t_i))-I_i(0)\|+\|I_i(0)\|\}.\end{aligned}$$

Using H_1, H_2, H_5 and H_7, we get

$$\begin{aligned}\left\|(Pu_\mu)(t)\right\|\leq M_0\|u_0\|+M_0K_1&\\+\int_0^1 M_0M_1M_2\Bigg[\|u_1\|+M_0\|u_0\|+M_0K_1&\\+\int_0^a M_0\left\{K_7\left(\left\|f\left(s,u(\gamma(s))\right)\right\|\right.\right.&\\\left.\left.+\left\|\int_0^s g\left(s,\tau,u(\delta(\tau))\right)d\tau)\right\|\right)+K_8\right\}ds&\\+M_0\sum_{i=1}^m(I_iC+\|I_i(0)\|)\Bigg]d\eta&\\+\int_0^t M_0\left\{K_7\left(\left\|f\left(s,u(\gamma(s))\right)\right\|\right.\right.&\\\left.\left.+\left\|\int_0^s g\left(s,\tau,u(\delta(\tau))\right)d\tau)\right\|\right)+K_8\right\}ds&\\+M_0\sum_{i=1}^m(I_iC+\|I_i(0)\|).&\end{aligned}$$

We have

$$\begin{aligned}\left\|(\mathrm{p}u_\mu)(t)\right\|\leq M_0\|u_0\|+M_0k_1&\\+M_0M_1M_2[\|u_1\|+M_0\|u_0\|+M_0k_1&\\+M_0k_7\theta+M_0k_8\mathrm{a}+M_0\xi]a+M_0k_7\theta&\\+M_0k_8\mathrm{a}+M_0\xi.&\end{aligned}$$

From assumption H_8, one gets $(\mathrm{p}u_\mu)(t)\leq C.$ thus p maps s_C into itself.

Now for $u, v\in s_C$, we have

$$\left\|\mathrm{p}u_\mu(t)-\mathrm{p}v_\mu(t)\right\|\leq I_1+I_2+I_3+I_4,$$

Where

$$I_1=\|R_{(\beta,u)}(t,0)u_0-R_{(\beta,v)}(t,0)u_0\|+\|R_{(\beta,u)}(t,0)h(u)-R_{(\beta,v)}(t,0)h(v)\|,$$

$$\begin{aligned}I_2&\\=\int_0^t&\left\{\left\|R_{(\beta,u)}(t,\eta)B\tilde{E}^{-1}[u_1-R_{(\beta,u)}(a,0)u_0\right.\right.\\&-R_{(\beta,u)}(a,0)h(u)\\&-\int_0^a R_{(\beta,u)}(a,s)\emptyset(s,f\left(s,u(\gamma(s))\right),\int_0^s g\left(s,\tau,u(\delta(\tau))\right)d\tau)ds\\&-\sum_{i=1}^m R_{(\beta,u)}(a,t_i)I_i(u(t_i))]-R_{(\beta,v)}(t,\eta)\,B\tilde{E}^{-1}[u_1\\&-R_{(\beta,v)}(a,0)u_0-R_{(\beta,v)}(a,0)h(v)\\&-\int_0^a R_{(\beta,v)}(a,s)\emptyset(s,f\left(s,v(\gamma(s))\right),\int_0^s g\left(s,\tau,v(\delta(\tau))\right)d\tau)ds\\&\left.\left.-\sum_{i=1}^m R_{(\beta,v)}(a,t_i)I_i(v(t_i))]\right\|\right\}d\eta,\end{aligned}$$

$$\begin{aligned}I_3&\\=\int_0^t&\left\|R_{(\beta,u)}(t,s)\emptyset(s,f\left(s,u(\gamma(s))\right),\int_0^s g\left(s,\tau,u(\delta(\tau))\right)d\tau)\right.\\&\left.-R_{(\beta,v)}(t,s)\emptyset(s,f\left(s,v(\gamma(s))\right),\int_0^s g\left(s,\tau,v(\delta(\tau))\right)d\tau)\right\|\end{aligned}$$

And

$$I_4=\sum_{i=1}^m\|R_{(\beta,u)}(t,t_i)I_i(u(t_i))-R_{(\beta,v)}(t,t_i)I_i(v(t_i))\|.$$

Applying lemma 3.1 and H_2, we get

$$\begin{aligned}I_1\leq\|R_{(\beta,u)}(t,0)u_0-R_{(\beta,v)}(t,0)u_0\|&\\+\|R_{(\beta,u)}(t,0)h(u)-R_{(\beta,v)}(t,0)h(u)\|&\\+\|R_{(\beta,v)}(t,0)h(u)-R_{(\beta,v)}(t,0)h(v)\|&\\\leq\{ka\|u_0\|+k_1ka+M_0k_2\}\max_{\tau\in S}\|u(\tau)-v(\tau)\|.&\end{aligned}$$

Also we apply lemma 3.1 and 3.2 H_1, H_2, H_5 and H_8, we obtain

$$I_2 \le a^2kM_1M_2\left\{\left\|2max\left(\left[u_1 - R_{(\beta,u)}(a,0)u_0 + R_{(\beta,u)}(a,0)h(u) - \int_0^a R_{(\beta,u)}(a,s)\emptyset\left(s, f\left(s,u(\gamma(s))\right), \int_0^s g\left(s,\tau,u(\delta(\tau))\right)d\tau\right)ds - \sum_{i=1}^m R_{(\beta,u)}(a,t_i)\{I_i(u(t_i)) - I_i(0) + I_i(0)\}\right], \left[u_1 - R_{(\beta,v)}(a,0)u_0 + R_{(\beta,v)}(a,0)h(v) - \int_0^a R_{(\beta,v)}(a,s)\varphi\left(s, f\left(s,v(\gamma(s))\right), \int_0^s g\left(s,\tau,v(\delta(\tau))\right)d\tau\right)ds - \sum_{i=1}^m R_{(\beta,v)}(a,t_i)\{I_i(v(t_i)) - I_i(0) + I_i(0)\}\right]\right)\right\|\right\}\max_{\tau\epsilon S}\|u(\tau) - v(\tau)\|$$

$$\le 2a^2kM_1M_2\{\|u_1\|Y + M_0(\|u_0\|Y + K_1 + K_7\theta + K_8a + \xi)\}\max_{\tau\epsilon S}\|u(\tau) - v(\tau)\|.$$

Again, Lemmas 3.1, $H_3 - H_6$ and H_8 imply that

$$I_3 \le \int_0^t \left\{\left\|R_{(\beta,u)}(t,s)\emptyset\left(s, f\left(s,u(\gamma(s))\right), \int_0^s g\left(s,\tau,v(\delta(\tau))\right)d\tau\right) - R_{(\beta,v)}(t,s)\varphi\left(s, f\left(s,u(\gamma(s))\right), \int_0^s g\left(s,\tau,v(\delta(\tau))\right)d\tau\right)\right\| + \left\|R_{(\beta,u)}(t,s)\emptyset\left(s, f\left(s,u(\gamma(s))\right), \int_0^s g\left(s,\tau,v(\delta(\tau))\right)d\tau\right) - R_{(\beta,v)}(t,s)\varphi\left(s, f\left(s,u(\gamma(s))\right), \int_0^s g\left(s,\tau,v(\delta(\tau))\right)d\tau\right)\right\|\right\}ds$$

$$\le K_a(K_7\theta + K_8a)\max_{\tau\epsilon S}\|u(\tau) - v(\tau)\| + M_0K_7\int_0^t\left\{\left\|f\left(s,u(\gamma(s)\right) - f\left(s,v(\gamma(s))\right)\right\| + \int_0^s\left\|g\left(s,\tau,u(\delta(\tau))\right) - g\left(s,\tau,v(\delta(\tau))\right)\right\|\right\}ds$$

$$\le K_a(K_7\theta + K_8a)\max_{\tau\epsilon S}\|u(\tau) - v(\tau)\| + M_0K_7\int_0^t\left\{K_5\sum_{p=1}^r\left\|u\left(\gamma_p(s)\right) - v\left(\gamma_p(s)\right)\right\|\left(\frac{\gamma_p'(s)}{c_p}\right) + K_3\sum_{p=1}^r\left\|u\left(\delta_p(s)\right) - v\left(C_p(s)\right)\right\|\left(\frac{\delta_q'(s)}{b_p}\right)\right\}ds$$

$$\le \{K_a(K_7\theta + K_8a + M_0k_{7\rho})\}\max_{\tau\epsilon S}\|u(\tau) - v(\tau)\|.$$

Now, form lemma 3.1, H_7 *and* H_8 , we have

$$I_3 \le \sum_{i=1}^m\{+\|-R_{(\beta,v)}(t,t_i)I_i(u(t_i))\|\}$$

$$\le \left\{k\sum_{i=1}^m(l_iC + \|I_i(0)\|)a + M_0\sum_{i=1}^m l_i\right\}\max_{\tau\epsilon S}\|u(\tau) - v(\tau)\|.$$

It follows from these estimations that

$$\|Pu_\mu(t) - Pv_\mu(t)\| \le \sum_{j=1}^4 I_j \le \sum_{i=1}^4\lambda_j\max_{\tau\epsilon S}\|u(\tau) - v(\tau)\|.$$

Therefore, P is a contraction mapping and hence there exists a unique fixed point $u\epsilon X$, such that $Pu(t) = u(t)$. Any fixed point of P is a mild solution of (1.1)-(1.3) on J which satisfies $u(a) = u_1$. Thus, system (1)-(3) is controllable on J.

4. Example

Consider the following fractional nonlocal impulsive integro-partial differential control system of the form

$$\frac{\partial^\beta}{\partial t^\beta}z(t,y) = \frac{\partial^2}{\partial y^2}z(t,y) + \mu(t,y) + k_0(y)\sin z(t,y) + k_1\int_0^t e^{-z(s,y)ds},$$

$$z(0,y) + \sum_{i=1}^m c_i\varphi(t_i,y) = z_0(y), \quad \varphi \in z, 0 \le y \le \pi,$$

$$z(t,0) = z(t,\pi) = 0, \quad t \in J = [0,b],$$

Where $0 < \alpha < 1, k_0(y)$ is continuous on $[0,\pi]$ and $c_i > 0, k_1 > 0$.

Let us take

$X = U = L^2[0,\pi], \quad Z = C([0,b],B_r), \quad B_r = \{y \in L^2[0,\pi]: \|y\| \le r\}$.

Put $x(t) = z(t,.)$ *and* $u(t) = \mu(t,.)$ *where* $\mu: J \times [0,\pi] \to [0,\pi]$ is continuous,

$$g(\varphi(t,.)) = \sum_{i=1}^m c_i\left(\varphi(t,.)\right)$$

and $f(t,x,Hx) = k_0(.)\sin z(t,.) + Hx, h(t,s,x) = k_1e^{-z(s,.)}$.

Let $A: D(A) \subset X \to X$ be the operator defined by $Az = z''$ with the domain

$D(A) = \{z \in X: z, z'\ absolutely\ continuous, z'' \in X, z(0) = z(\pi) = 0\}$.

Then

$$Az = \sum_{n=1}^\infty n^2(z,z_n)z_n, \quad z \in D(A)$$

Where $z_n(y) = \sqrt{2/\pi}\sin ny, n = 1,2,3,...$ is the orthogonal set of eigenvector of A. it is well known that A is the infinitesimal generator of an analytic semigroup $T(t), t \ge 0$ in X and is given by

$$T(t)z = \sum_{n=1}^\infty e^{-n^2t}(z,z_n)z_n, \quad z \in X.$$

With this choice of $A, f, g, H\ and\ B = I$, Assume that the operator $W: L^2(J, U)/\ker W \to X$ defined by

$$Wu = \frac{1}{\Gamma(\beta)} \int_0^b (b-s)^{\beta-1} T(b-s) u(s) ds$$
$$= \frac{1}{\Gamma(\beta)} \sum_{n=1}^{\infty} (b - s)^{\beta-1} e^{-n^2(b-s)} (u(s), z_n) z_n\, ds$$

has an inverse operator.

5. Conclusion

In this article, the controllability result for a class of fractional evolution nonlocal impulsive quasilinear multi-delay integro-differential systems in a Complete vector space has been considered. A new set of sufficient conditions are derived for our main result by using the theory of fractional calculus, fixed point technique and (β, u)-resolvent family (a new concept).

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