Reflected Backward Stochastic Differential Equations Driven by Countable G-Brownian Motion

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Abstract: In this paper, we deal with a new-fangled class of reflected backward stochastic differential equations driven by G-Brownian motion, the existence and uniqueness of the backward stochastic differential equations are obtained by way of snell covering and fixed point theorem.

Keywords: Levy process, fixed point theorem, monotone convergence theorem, Gronwall's inequality, Lipchitz condition.

1. Introduction

The nonlinear BSDE were introduced by Pardoux and Peng [1] who proved the existence and uniqueness of the result in the Liechtz condition for benevolent the probabilistic elucidation of the semi linear parabolic partial differential equations. Firstly, studied the backward doubly stochastic differential equations which are driven by two kinds of G-Brownian motions. Then, Boufoussi et. al.[10] established BSDE and semi linear stochastic partial differential equations with a Neumann boundary condition. We deliberate the reflected backward doubly stochastic differential equations driven by Levy process and the equations driven by finite G-Brownian motion.

2. Notations

Q is the positive constant. Throughout the paper (Λ, ℂ, ℙ) is complete Probability space prepared with the ordinary filtration {ℂq}q≥0 satisfying the usual conditions. {αq(q)}q=1 are mutually independent one dimensional standard Brownian motion on the probability space. ℳG(q) is the standard G-Brownian motion on ℋ2 which is independent of αq(q).

Assume that ℂq = (Vq ⊂, ℂq, q)  V ℂq G V N ... (1)

Where any process {μq}q r,q = δ{μp − μr: r ≤ p ≤ q}

An q = Anq and N denotes the class of null sets of ℂ.

Let us initiate a few spaces:

(i) ℋ2 = {φq :q≤q} an ℂq gradually measurable ℋ valued process such that E f|φq| 2 dq ≤ 2 < ∞

(ii) ℂ = {ηq :q≤q} ℂq gradually measurable ℋd valued continuous process such that E sup|ηq| 2 < ∞

(iii) ℂ = {kq :q≤q} an ℂq adopted continuous increased process such that k0 = 0, E[k0] 2 < ∞.

With the preceding measures, we deliberate the following RBSDES:

Xq = ζ + ∩q f(r, Xr, Yr) dr + ∩q g(q, r, Xr, Yr) da(r) - ∩q Yr dW G(r) + Wq − Wq, 0 ≤ q ≤ Q ... (2)

Where f: Λ × [0, Q] × ℋ × ℋd → ℋ and g: Λ × [0, Q] × ℋ × ℋd → ℋ

Definition 1: A Solution of (2) is a triple of ℋ × ℋd → ℋ value process (Xq, Yr, kq) which satisfies (2)and

(i) Xr ≥ Rr;

(ii) (Xq, Yr, kq) ∈ ℋ2 × ℋ2 × ℋ2

(iii) kq is a incessant and growing process with k0 = 0 and f|q (Xq − Rq) d |kq = 0

To facilitate get the solution of (2), we intend the following assumptions:

(a) ζ is an ℂQ assessable square integrable random variable;

(b) the obstacle {Rq: 0 ≤ q ≤ Q} is an ℂQ progresssive measurable incessant real valued process which satisfies E sup|q0(Rq) 2 < ∞. We always assume that Rq ≤ ζ;

(c) f(, x, y) and g(, x, y) are two progressive measurable functions such that , for any q ∈ [0, Q], x, x ∈ ℋ, y, y ∈ ℋ, (2a)f(r, ,...) is incessant and |f(r, x, y)| ≤ M(1 + |x| + |y|);

(2b) |f(q, o, a)| 2 dq < ∞.

(2c)|f(r, x, y) − f(r, x, y)| 2 ≤ L|x − x| 2 + |y1 − y2| 2 + g(r, x, y1) − g(r, x, y2)| 2 ≤ L|x − x| 2 + β1|y1 − y2| 2
where $\mathcal{M}, P, P_i$ and $\beta_i$ are non-negative constants with $\sum_{i=1}^{\infty} P_i < \infty$, and
\[
\beta = \sum_{i=1}^{\infty} \beta_i < 1
\]

3. Main result

In order to get the solution of (2), we consider the following RBSDEs driven by the finite Brownian motions:
\[
X_q = \zeta + \int_{0}^{q} f(r, X_r, Y_r) dr + \sum_{i=1}^{\infty} \int_{0}^{\infty} g_i(r, X_r, Y_r) d\alpha_i(r) - J^0_q Y_r dW^G(r) + \mathbb{K}_Q - \mathbb{K}_q, 0 \leq q \leq Q...(3)
\]

Firstly, we consider a special case of (3), the functions $f$ and $g$ do not depend on $(X, Y)$,
\[
X_q = \zeta + \int_{0}^{q} f(r) dr + \sum_{i=1}^{\infty} \int_{0}^{\infty} g_i(r) d\alpha_i(r) - J^0_q Y_r dW^G(r) + \mathbb{K}_Q - \mathbb{K}_q, 0 \leq q \leq Q...(4)
\]

Theorem 2:

Assume that (i) –(ii) $f \in \mathcal{H}^2, g \in \mathcal{H}^2$ then there exists a triple $(X_r, Y_r, \mathbb{K}_q)_{0 \leq q \leq Q} \in \mathcal{H}^2 \times \mathcal{H}^2 \times \mathcal{H}^2$

Proof:

Let $\mathcal{F}_q = X_q^{\mathcal{C}a\kappa^G} V(Y_q^{\mathcal{C}a\kappa^G}) ...(5)$

Define $\mu = \{ \mu_q \}_{0 \leq q \leq Q}$ as

$\mu_q = \zeta 1_{(0 \leq q \leq Q)} + R_q 1_{q \leq Q} \int_{0}^{q} f(r) dr + \sum_{i=1}^{n} \int_{0}^{q} g_i(r) d\alpha_i(r) ...(6)$

Then, $\mu$ is $\mathcal{F}_q$ adapted continuous process; furthermore;

$\sup_{0 \leq q \leq Q} |\mu_q| \in \ell^2(\Lambda) ...(7)$

The snell envelope of $\mu$ is given by

$R_q(\mu) = \text{ess sup}_{\psi \in \Psi} \mathbb{E}[\psi\text{ such that } \mathcal{F}_q] ...(8)$

Where $\Psi$ is set of all $\mathcal{F}_q$ stopping time such that $0 \leq q \leq Q$

By the definition of $\mu$ we can deduce that

$\mathbb{E}\left[\sup_{0 \leq q \leq Q} |R_q(\mu)|^2 \right] < \infty ...(9)$

Due to the doob-meyer decomposition, we have

$R_q(\mu) = \mathbb{E}[\zeta + \int_{0}^{q} f(r) dr + \sum_{i=1}^{n} \int_{0}^{q} g_i(r) d\alpha_i(r) + \mathbb{K}_Q: \mathcal{F}_q] - \mathbb{K}_q ...(10)$

Where $\{\mathbb{K}_q\}_{0 \leq q \leq Q}$ is a adopted, continuous and non decreasing process such that $\mathbb{K}_0 = 0$ and $\mathbb{E} \mathbb{K}_q^2 < \infty$ so we have

$\mathbb{E}\left[\sup_{0 \leq q \leq Q} \mathbb{E}[\zeta + \int_{0}^{q} f(r) dr + \sum_{i=1}^{n} \int_{0}^{q} g_i(r) d\alpha_i(r) + \mathbb{K}_Q: \mathcal{F}_q] \right]^2 ...(11)$

Martingale representation theorem yields that there exists $\mathcal{F}_q$ progressive measurable process $(Y_q) \in \mathcal{H}^d$
if we define $\psi'_M(a)/a = 2$ when $a=0$, then $0 \leq \psi'_M(\bar{x}_r)/\bar{x}_r \leq 2$. Applying this formula to $e^{aq} \psi_M(\bar{x}_r)$, we have

$$e^{aq} \psi_M(\bar{x}_r) + \alpha \int_q^\infty e^{ar} \psi_M(\bar{x}_r) \, dr + \int_q^\infty e^{ar} \mathbb{1}_{[-M \leq \bar{x}_r \leq M]} |\bar{y}_r|^2 \, dr$$

$$= \int_q^\infty e^{aq} \psi_M(\bar{x}_r) \left( f(r, \bar{x}_r, \bar{y}_r) - f(r, \bar{x}_r', \bar{y}_r') \right) \, dr + \sum_{i=1}^n \int_q^\infty e^{aq} \psi_M(\bar{x}_r) \left( g_i(r, \bar{x}_r, \bar{y}_r) - g_i(r, \bar{x}_r', \bar{y}_r') \right) \, dr$$

$$- \int_q^\infty e^{aq} \psi_M(\bar{x}_r) \, d\omega G(r) + e^{aq} \psi'_M(\bar{x}_r) \, d\|k_r - d\|k'_r \ldots (19)$$

Taking expectation on both sides of (19) and noticing that

$$\int_q^\infty e^{aq} \psi_M(\bar{x}_r) \, d\|k_r - d\|k'_r \leq 0,$$ we have

$$E e^{aq} \psi_M(\bar{x}_r) + E\alpha \int_q^\infty e^{ar} \psi_M(\bar{x}_r) \, dr$$

$$\leq E \int_q^\infty e^{ar} \psi_M(\bar{x}_r) \left( f(r, \bar{x}_r, \bar{y}_r) - f(r, \bar{x}_r', \bar{y}_r') \right) \, dr$$

$$+ \sum_{i=1}^n \int_q^\infty e^{ar} \mathbb{1}_{[-M \leq \bar{y}_r \leq M]} |g_i(r, \bar{x}_r, \bar{y}_r)| \, dr$$

$$- \sum_{i=1}^n \int_q^\infty e^{ar} \psi_M(\bar{x}_r) \left( g_i(r, \bar{x}_r, \bar{y}_r) - g_i(r, \bar{x}_r', \bar{y}_r') \right) \, dr$$

$$\leq 2E \int_q^\infty e^{aq} \psi_M(\bar{x}_r) \left( f(r, \bar{x}_r, \bar{y}_r) - f(r, \bar{x}_r', \bar{y}_r') \right) \, dr$$

$$+ \sum_{i=1}^n \int_q^\infty e^{ar} \mathbb{1}_{[-M \leq \bar{y}_r \leq M]} |g_i(r, \bar{x}_r, \bar{y}_r)| \, dr$$

$$\leq 2 \sum_{i=1}^n \int_q^\infty e^{ar} \mathbb{1}_{[-M \leq \bar{y}_r \leq M]} |\bar{y}_r|^2 \, dr + r^2 \sum_{i=1}^n \int_q^\infty e^{ar} \mathbb{1}_{[-M \leq \bar{y}_r \leq M]} \left( |\bar{x}_r|^2 + \alpha \right) \, dr$$

Let $\lambda = \frac{2p}{1 - p}$, $\beta = 2(\sum_{i=1}^n P_i + (1 - \alpha)/2)/(1 + \alpha)$, $\delta = \lambda + P_i$, and $M \to \infty$; we have

$$P E \int_q^\infty e^{ar} \mathbb{1}_{[\bar{x}_r, \bar{x}_r']^2} \, dr + E \int_q^\infty e^{ar} |\bar{y}_r - \bar{y}_r'|^2 \, dr$$

$$\leq \frac{1 + \alpha}{2} \left( \int_q^\infty e^{ar} (P|\bar{x}_r - \bar{x}_r'|^2 + |\bar{y}_r - \bar{y}_r'|^2) - 2 \right) \ldots (21)$$

that is $|X_r, Y_r, K_r|^2 \leq (1 - \alpha)\|x_r, y_r, k_r\|^2 \ldots (22)$

It follows that $\lambda$ is a strict contradiction on with the norm $\| \cdot \|_{\lambda}$, where $\alpha$ is defined as above. Then $\lambda$ has a fixed point $(X, Y, K)$ which is the unique solution of (4) from the Burholder–Davis–Gundy inequality.

**Theorem 4:**
Under the conditions of (a)-(c), they exist a unique solution $(X_q, Y_q, K_q)$ for every $q \geq 0$.

Proof (existence) by the theorem 3, for any $n \geq 1$, there exists a unique solution of (3) denoted by $(X_q^n, Y_q^n, K_q^n)$.

$$Y_q^n = \int_q^\infty f(r, X_q^n, Y_q^n) \, dr + \sum_{i=1}^n \int_q^\infty g_i(r, X_q^n, Y_q^n) \, d\alpha_i(r) - \int_q^\infty Y_q^n \, dW^G(r) + d\|k_q^n - \|k_q^n\|s \ldots (23)$$

In the following parts, we will claim that $(X_q^n, Y_q^n, K_q^n)$ is a Cauchy sequence in $\mathcal{D} \times \mathcal{D} \times \mathcal{H}$. Without loss of generality, we let $n < m$ applying general formula to $|X_q^n - X_q^m|^2$, we have

$$|X_q^n - X_q^m|^2 + \int_q^\infty |Y_q^n - Y_q^m|^2 \, dr$$

$$= 2 \int_q^\infty (X_q^n - X_q^m)(f(r, X_q^n, Y_q^n) - f(r, X_q^m, Y_q^m)) \, dr$$

$$+ \sum_{i=1}^n \int_q^\infty g_i(r, X_q^n, Y_q^n) - g_i(r, X_q^m, Y_q^m) \, d\alpha_i(r)$$

$$- 2 \int_q^\infty (X_q^n - X_q^m)(Y_q^n - Y_q^m) \, dW^G(r) + 2 \int_q^\infty (X_q^n - X_q^m) \, d\|k_q^n - \|k_q^m\|s \ldots (24)$$

Taking expectation on both sides of (24) and nothing that

$$\int_q^\infty (X_q^n - X_q^m) \, d\|k_q^n - \|k_q^m\|s \leq 0$$

we obtain

$$E|X_q^n - X_q^m|^2 \leq 2E \int_q^\infty f(r, X_q^n, Y_q^n) - f(r, X_q^m, Y_q^m) \, dr$$

$$+ \sum_{i=1}^n \int_q^\infty g_i(r, X_q^n, Y_q^n) - g_i(r, X_q^m, Y_q^m) \, d\alpha_i(r)$$

$$= 2 \int_q^\infty \mathbb{E} \left( X_q^n - X_q^m \right)^2 + \mathbb{E} |Y_q^n - Y_q^m|^2 \, dr$$

$$\leq 2E \int_q^\infty \mathbb{E} \left( X_q^n - X_q^m \right)^2 + \mathbb{E} |Y_q^n - Y_q^m|^2 \, dr$$

By (c) and elementary inequality $2xy \leq ax^2 + (\frac{b}{a})y^2$, $a > 0$, we obtain

$$E|X_q^n - X_q^m|^2 \leq 2 \beta E \int_q^\infty \mathbb{E} \left( X_q^n - X_q^m \right)^2 \, dr + (1 - \beta) \int_q^\infty \mathbb{E} |Y_q^n - Y_q^m|^2 \, dr$$

$$+ (1 - \beta)/2 E \int_q^\infty \mathbb{E} |Y_q^n - Y_q^m|^2 \, dr$$

Furthermore

$$E|X_q^n - X_q^m|^2 \leq 1 - \beta/2 E \int_q^\infty \mathbb{E} |Y_q^n - Y_q^m|^2 \, dr \leq$$
By Gronwall’s inequality and Burkholder-Davis-Gundy inequality, we have:

\[ E \left| \int_0^Q f(r, X_r, Y_r) - f((r, X_r^n, Y_r^n)) \right|^2 dr \leq QE \left| \int_0^Q f(r, X_r, Y_r) - f((r, X_r^n, Y_r^n)) \right|^2 dr \to 0 \quad \ldots (29) \]

From (27), we know that:

\[ E \int_0^Q |X_n^2 - X_m^2| \to 0 \quad \ldots (30) \]

and:

\[ \left| X_q^n - X_q^m \right| \leq \left| X_q^1 \right| + \sum_{k=1}^n \left| X_q^{k+1} - X_q^k \right| \leq \left| X_q^1 \right| + \sum_{k=1}^n \left| X_q^{k+1} - X_q^k \right| \quad \ldots (32) \]

Then we have:

\[ \left( \left| X_q^1 \right| + \sum_{k=1}^n \left| X_q^{k+1} - X_q^k \right| \right)^2 dq \leq \left( \left| X_q^1 \right| + \sum_{k=1}^n \left| X_q^{k+1} - X_q^k \right| \right)^2 dq \leq \left( \sum_{k=1}^n \left| X_q^{k+1} - X_q^k \right| \right)^2 dq \quad \ldots (33) \]

By Lebesgue convergence theorem, we deduce that:

\[ \left( \sum_{k=1}^n \left| X_q^{k+1} - X_q^k \right| \right)^2 dq \leq \left( \sum_{k=1}^n \left| X_q^{k+1} - X_q^k \right| \right)^2 dq \leq \left( \sum_{k=1}^n \left| X_q^{k+1} - X_q^k \right| \right)^2 dq \quad \ldots (34) \]

Applying Lebesgue convergence theorem, we deduce that:

\[ \left( \sum_{k=1}^n \left| X_q^{k+1} - X_q^k \right| \right)^2 dq \leq \left( \sum_{k=1}^n \left| X_q^{k+1} - X_q^k \right| \right)^2 dq \leq \left( \sum_{k=1}^n \left| X_q^{k+1} - X_q^k \right| \right)^2 dq \quad \ldots (34) \]

Uniqueness: Let \( (X_q^k, Y_q^k, k_q^k) \) be two solutions of (2), \( X_q^1(X_q^1, Y_q^1) = Y_q^2(X_q^2, Y_q^2) \). Applying Itô formula to:

\[ e^{aq} \psi_{M}(Y_q) + \alpha \int_q^Q e^{ar} \psi_{M}(Y_r) dr + \int_q^Q e^{ar} 1_{\{-M \leq Y_r \leq M\}} Y_r^2 dr \]

we have:

\[ q \int e^{ar} \psi_{M}(X_q) (f(r, X_r^2, Y_r^2) - f(r, X_r^2, Y_r^2)) dr + \sum_{k=1}^\infty \int_q^Q e^{ar} 1_{\{-M \leq Y_r \leq M\}} g_i((r, X_r^1, Y_r^1) - \ldots (35) \]

Taking expectation of (35),

\[ Ee^{aq} \psi_{M}(X_q) + \alpha E \int_q^Q e^{ar} \psi_{M}(Y_r) dr + \int_q^Q e^{ar} 1_{\{-M \leq Y_r \leq M\}} Y_r^2 dr \]

\[ \leq 2E \int_q^Q e^{ar} \psi_{M}(X_q) (f(r, X_r^1, Y_r^1) - f(r, X_r^2, Y_r^2)) dr + \sum_{k=1}^\infty \int_q^Q e^{ar} 1_{\{-M \leq Y_r \leq M\}} g_i((r, X_r^1, Y_r^1) - \ldots (36) \]

Let \( \lambda \to \infty \) and applying monotone convergence theorem, we have:

\[ E \int_q^Q e^{ar} \psi_{M}(Y_r) dr + \alpha \int_q^Q e^{ar} \psi_{M}(Y_r) dr + \int_q^Q e^{ar} 1_{\{-M \leq Y_r \leq M\}} Y_r^2 dr \leq \alpha \int_q^Q e^{ar} \psi_{M}(Y_r) dr + \ldots (37) \]

When \( \alpha \) is taken sufficiently large, we have:

\[ Y_q \leq \ldots (37) \]

It follows that \( \lambda \) is a strict contradiction with the norm \( \| Y \|_{\alpha} \) where \( \alpha \) is defined as above. Then \( \lambda \) has a fixed point \( (X_q, Y_q, k_q) \) which is the unique solution of (4) from the Burkholder-Davis–Gundy inequality.

**Theorem 4:**

Under the conditions of (a)-(c), there exists a unique solution \( (X_q, Y_q, k_q) \) for all \( q \in \mathbb{R} \).

Proof (existence) by the theorem 3, for any \( n \geq 1 \), there exists a unique solution of (3), denoted by \( (X_q^n, Y_q^n, k_q^n) \),

\[ Y_q^n = z + \int_q^Q f(r, X_r^n, Y_r^n) dr + \sum_{k=1}^n \int_q^Q g_i(r, X_r^n, Y_r^n) d\alpha(r) \]

In the following parts, we will claim that \( (X_q^n, Y_q^n, k_q^n) \) is a Cauchy sequence in \( \mathbb{R}^2 \) without loss of generality. Let \( n < m \). Applying general formula to \( X_q^n - X_q^m \), we have:

\[ |X_q^n - X_q^m|^2 + \int_q^Q |Y_r^n - Y_r^m|^2 dr \]

\[ = 2 \int_q^Q (X_q^n - X_q^m) (f(r, X_r^n, Y_r^n) - f(r, X_r^m, Y_r^m)) dr \]

\[ + \sum_{k=1}^\infty \int_q^Q e^{ar} 1_{\{-M \leq Y_r \leq M\}} g_i((r, X_r^1, Y_r^1) - \ldots (38) \]

By Gronwall’s inequality and Burkholder-Davis-Gundy inequality, we have:

\[ E \left| \int_0^Q f(r, X_r, Y_r) - f((r, X_r^n, Y_r^n)) \right|^2 dr \leq QE \left| \int_0^Q f(r, X_r, Y_r) - f((r, X_r^n, Y_r^n)) \right|^2 dr \to 0 \quad \ldots (29) \]

From (27), we know that:

\[ E \int_0^Q |X_n^2 - X_m^2| \to 0 \quad \ldots (30) \]

and:

\[ \left| X_q^n - X_q^m \right| \leq \left| X_q^1 \right| + \sum_{k=1}^n \left| X_q^{k+1} - X_q^k \right| \leq \left| X_q^1 \right| + \sum_{k=1}^n \left| X_q^{k+1} - X_q^k \right| \quad \ldots (32) \]

Then we have:

\[ \left( \left| X_q^1 \right| + \sum_{k=1}^n \left| X_q^{k+1} - X_q^k \right| \right)^2 dq \leq \left( \left| X_q^1 \right| + \sum_{k=1}^n \left| X_q^{k+1} - X_q^k \right| \right)^2 dq \leq \left( \sum_{k=1}^n \left| X_q^{k+1} - X_q^k \right| \right)^2 dq \quad \ldots (33) \]

By Lebesgue convergence theorem, we deduce that:

\[ \left( \sum_{k=1}^n \left| X_q^{k+1} - X_q^k \right| \right)^2 dq \leq \left( \sum_{k=1}^n \left| X_q^{k+1} - X_q^k \right| \right)^2 dq \leq \left( \sum_{k=1}^n \left| X_q^{k+1} - X_q^k \right| \right)^2 dq \quad \ldots (34) \]
\[ + \sum_{i=m+1}^{\infty} \int_{q}^{r} |g_i(r, X_r, Y_r)| \gamma (r, X_r, m, Y_r) |^2 dr \]
\[ - 2 \sum_{i=m+1}^{\infty} \int_{q}^{r} (X_r - X_r m) (g_i(r, X_r, m, Y_r) - g_i(r, X_r, m, Y_r) m) d \alpha (r) \]
\[ - 2 \frac{\alpha}{\gamma} \int_{q}^{r} (X_r - X_r m) (Y_r - Y_r m) d \gamma (r) \]
\[ + 2 \int_{q}^{r} (X_r - X_r m) (d \alpha (r) - d \alpha (r)) \ldots (24) \]

Taking expectation on both sides of (24) and nothing that
\[ \int_{q}^{r} (X_r - X_r m) (d \alpha (r) - d \alpha (r)) \leq 0, \]
we obtain
\[ E |X_r^n - X_r m|^2 + E |Y_r - Y_r m|^2 dr \]
\[ \leq 2 E \int_{q}^{r} (X_r^n - X_r m) (f(r, X_r, m, Y_r) - f(r, X_r, m, Y_r) m) dr \]
\[ + \sum_{i=m+1}^{\infty} E \int_{q}^{r} y_i (r, X_r, m, Y_r) - g_i (r, X_r, m, Y_r) |^2 dr \]

By (c) and elementary inequality \[ 2xy \leq \alpha x^2 + (\frac{1}{\alpha}) y^2, \alpha > 0, \]
we obtain
\[ E |X_r^n - X_r m|^2 + E |Y_r - Y_r m|^2 dr \]
\[ \leq 2 \left( \frac{1}{1 - \beta} \right) \int_{q}^{r} (X_r^n - X_r m) |^2 dr \]
\[ + (1 - \beta) / 2 \int_{q}^{r} |Y_r - Y_r m|^2 dr + \beta \int_{q}^{r} |Y_r - Y_r m|^2 dr \]
\[ + \sum_{i=m+1}^{\infty} P_i E \int_{q}^{r} (X_r^n - X_r m)^2 dr |X_r^n - Y_r m|^2 \] \[ \ldots (26) \]

Furthermore,
\[ E |X_r^n - X_r m|^2 + 1 - \beta / 2 \quad E \int_{q}^{r} |Y_r - Y_r m|^2 dr \leq \]
\[ P_{i_c} E |X_r^n - X_r m|^2 dr \ldots (27) \]

Where \[ P_{i_c} = (\frac{2 \beta}{1 - \beta})^{1 - \beta} + \frac{1}{\gamma} \sum_{i=m+1}^{\infty} P_i \]

By Gronwall’s inequality and Burholder-Davis-Gundy inequality \[ E \sup_{0 \leq q \leq r} |Y_r - Y_r m|^2 dr \rightarrow 0 \ldots (28) \]

Denote the limit of \( (X_r^n, Y_r^n, \mathbb{K}_r^n) \) by \( (X_r, Y_r, \mathbb{K}_r) \); we will show that \( (X_r, Y_r, \mathbb{K}_r) \) satisfies (2). If it is necessary, we can choose a subsequence of (3). By Holder’s inequality,
\[ E \int_{q}^{r} f(r, X_r, Y_r) - f((r, X_r, m, Y_r)) \]
\[ \leq QE \int_{q}^{r} f(r, X_r, Y_r) - f((r, X_r, m, Y_r)) |^2 dr \rightarrow 0 \ldots (29) \]

From (27), we know \[ E f_0 |X_r^n - X_r m|^2 \rightarrow 0 \ldots (30) \] and
\[ X_r^n \rightarrow X_r, \quad \Rightarrow \quad E \int_{q}^{r} \frac{|X_r^{n+1} - X_r m|^2}{2} \leq \frac{1}{2^n} \ldots (31) \] for any n,
\[ |X_r^n| \leq |X_r^1| + \sum_{k=1}^{n} |X_r^{k+1} - X_r^k| \leq |X_r^1| + \sum_{k=1}^{\infty} |X_r^{k+1} - X_r^k| \ldots (32) \]

Then, we have
\[ E \int_{q}^{r} \sup \frac{|X_r^n|^2}{n} dq \]
\[ \leq E \int_{q}^{r} \left( |X_r^1| + \sum_{k=1}^{\infty} |X_r^{k+1} - X_r^k| \right)^2 dq \]
\[ \leq \sqrt{E \int_{q}^{r} |X_r^1|^2 dq} + \sum_{k=1}^{\infty} \frac{1}{2^k} \int_{q}^{r} |X_r^1|^2 dq \]
\[ \leq \sqrt{E \int_{q}^{r} |X_r^1|^2 dq} + |X_r|^2 + |X_r|^2 + |Y_r|^2 dr < \infty \]
\[ \ldots (34) \]

Applying Lebesgue convergence theorem, we deduce that \( (X_r, Y_r, \mathbb{K}_r) \) is the solution of (2) by the continuity of the functions \( f \) and \( g \).

Uniqueness: Let \( \{X_r^k, Y_r^k, \mathbb{K}_r^k \} (k = 1, 2) \) be two solutions of (2). \( \bar{X}_r = X_r^1 - X_r^2, \bar{Y}_r = Y_r^1 - Y_r^2 \). We apply Ito formula to
\[ e^{aq} \theta \psi_M (\bar{X}_r) + \alpha \int_{q}^{r} e^{ar} \psi_M (\bar{Y}_r) dr \]
\[ + \int_{q}^{r} e^{ar} a \psi_M (\bar{Y}_r) dr \]
\[ = \int_{q}^{r} e^{ar} \psi_M (\bar{X}_r) (f(r, X_r^1, Y_r^1) - f(r, X_r^2, Y_r^2)) dr \]
\[ \sum_{k=1}^{\infty} \int_{q}^{r} e^{ar} a \psi_M (\bar{X}_r) \left( g_1 (r, X_r^1, Y_r^1) - g_1 (r, X_r^2, Y_r^2) \right) d \alpha (r) \]
\[ - \int_{q}^{r} e^{ar} \psi_M (\bar{X}_r) Y_r dq \}
\[ + \int_{q}^{r} e^{ar} \psi_M (\bar{X}_r) (d \mathbb{K}_r - d \mathbb{K}_r) \ldots (35) \]

Taking expectation of (35),
\[ E e^{aq} \psi_M (\bar{X}_r) + \alpha E \int_{q}^{r} e^{ar} \psi_M (\bar{Y}_r) dr \]
\[ + \int_{q}^{r} e^{ar} a \psi_M (\bar{Y}_r) dr \]
\leq 2 E \int_{q}^{r} e^{ar} \psi_M (\bar{X}_r) (f(r, X_r^1, Y_r^1) - f(r, X_r^2, Y_r^2)) dr \]
\[ + \sum_{k=1}^{\infty} E \int_{q}^{r} e^{ar} a \psi_M (\bar{X}_r) (d \mathbb{K}_r - d \mathbb{K}_r) \ldots (35) \]
\[ g_i((r, x_r^2, y_r^2)) \leq \left( \frac{2p}{1 - \sum_{k=1}^{\infty} \beta_l} + \sum_{k=1}^{\infty} P_l \right) E \int_q^0 e^{\alpha r} \psi' \mathcal{M}(X_r) \, dr + \frac{1 + \sum_{k=1}^{\infty} \beta_l}{2} E \int_q^0 e^{\alpha r} \left| Y_r \right|^2 \, dr \]  

Let \( \mathcal{M} \to \infty \) and applying monotone convergence theorem, we have

\[ E e^{\alpha q} \left| X_q \right|^2 + \left( \alpha - \frac{2p}{1-\beta} - \sum_{k=1}^{\infty} P_l \right) \times E \int_q^0 e^{\alpha r} \left| X_r \right|^2 \, dr + \frac{1-\beta}{2} E \int_q^0 e^{\alpha r} \left| Y_r \right|^2 \, dr \leq 0 \quad (37) \]

When \( \alpha \) is taken sufficiently large, we have \( Y_q = 0 \).

### 4. Conclusion

This paper has some concepts of reflected backward stochastic differential equations driven by countable G-Brownian motions. The existence and uniqueness of RBSDEs are obtained.

### References


