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Reflected Backward Stochastic Differential Equations Driven by Countable G-Brownian Motion

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***Abstract*: In this paper, we deal with a new-fangled class of reflected backward stochastic differential equations driven by G-Brownian motion, the existence and uniqueness of the backward stochastic differential equations are obtained by way of snell covering and fixed point theorem.**

***Keywords*: Levy process, fixed point theorem, monotone convergence theorem, Gronwall's inequality, Lipchitz condition.**

1. Introduction

The nonlinear BSDE were introduced by Pardoux and Peng [1] who proved the existence and uniqueness of the result in the Lipchitz condition for benevolent the probabilistic elucidation of the semi linear parabolic partial differential equations. Firstly, studied the backward doubly stochastic differential equations which are driven by two kinds of G- Brownian motions. Then, Boufoussi et. al.[10] established BSDE and semi linear stochastic partial differential equations with a Neumann boundary condition. We deliberate the reflected backward doubly stochastic differential equations driven by Levy process and the equations driven by finite G-Brownian motion.

2. Notations

Q is the positive constant. Throughout the paper $(\Lambda, \mathcal{K}, \mathbb{p})$ is complete Probablity space prepared with the ordinary filtration $\{\mathcal{K}_q\}_{q\geq 0}$ satisfying the usual conditions. $\{\alpha_i(q)\}_{i=1}^{\infty}$ are mutually independent one dimensional standard Brownian motion on the probability space. $\mathbb{W}^G(\mathrm{q})$ is the standard G-Brownian motion on $\mathcal{R}^d$ which is independent of $\alpha_i(q)$.

Assume that $\mathcal{K}_q = \left(\vee_{i=1}^{\infty} \mathcal{K}_{q,Q}^{\alpha_i}\right) \vee \mathcal{K}_q^{\mathbb{W}^G} \vee \mathbb{N}$ …(1)

Where any process $\{\mu_q\}\mathcal{K}_{r,q}^{\mu} = \delta\{\mu_p - \mu_r : r \leq p \leq q\}$

$\mathcal{K}_q^{\mu} = \mathcal{K}_{0,q}^{\mu}$ and $\mathbb{N}$ denotes the class of $\mathbb{p}$ null sets of $\mathcal{K}$.

Let us initiate a few spaces:

(i)$\mathcal{H}^2 = \{(\phi_q)_{0\leq q\leq Q}$ an $\mathcal{K}_q$ gradually measurable $\mathcal{R}$ valued process such that $\mathbb{E}\int_0^Q |\phi_q|^2\, dq\mathcal{H}^2 < \infty\}$

(ii) $\mathcal{I}^2 = \{(\eta_q)_{0\leq q\leq Q}$; $\mathcal{K}_q$ gradually measurable $\mathcal{R}^d$ valued continuous process such that $\mathbb{E}(sup_{0\leq q\leq Q}|\eta_q|^2 < \infty\}$

(iii)$\mathcal{J}^2 = \{(\mathbb{K}_q)_{0\leq q\leq Q}$ an $\mathcal{K}_q$ adopted continuous increased process such that $\mathbb{K}_0 = 0, \mathbb{E}[\mathbb{K}_q]^2 < \infty\}$

with the preceding measures, we deliberate the following RBSDES:

$X_q = \zeta + \int_q^Q f(r, X_r, Y_r)\, dr + \sum_{i=1}^{\infty} \int_q^Q g_i(r, X_r, Y_r) d\alpha_i(r)$- $\int_q^Q Y_r dW^G(r) + \mathbb{K}_Q - \mathbb{K}_q, 0 \leq q \leq Q$…(2)

where $f{:}\Lambda \times [0,Q] \times \mathcal{R} \times \mathcal{R}^d \to \mathcal{R}$ and $g_i{:}\Lambda \times [0,Q] \times \mathcal{R} \times \mathcal{R}^d \to \mathcal{R}$

Definition 1: A Solution of (2) is a triple of $\mathcal{R} \times \mathcal{R}^d \to \mathcal{R}_+$ value process $(X_r, Y_r, \mathbb{K}_q)_{0\leq q\leq Q}$ which satisfies (2)and

(i) $X_r \geq R_r$;
(ii) $(X_r, Y_r, \mathbb{K}_q)_{0\leq q\leq Q} \in \mathcal{I}^2 \times \mathcal{J}^2 \times \mathcal{A}^2$
(iii) $\mathbb{K}_q$ is a incessant and growing process with $\mathbb{K}_0 = 0$ and $\int_0^Q (X_r - R_r) d\,\mathbb{K}_q = 0$

To facilitate get the solution of (2), we intend the following assumptions:

(a) ζ is an $\mathcal{K}_Q$ assessable square integrable random variable;
(b) the obstacle $\{R_r : 0 \leq q \leq Q\}$is an $\mathcal{K}_Q$ progressive measurable incessant real valued process which satisfies $\mathcal{E}sup_{0\leq q\leq Q}(R_r)^2 < \infty$. we always assume that $R_Q \leq \zeta$;
(c) $f(.,x,y)$ and $g_i(.,x,y)$ are two progressive measurable functions such that , for any $q \in [0,Q], x_1, x_2 \in \mathcal{R}, y_1, y_2 \in \mathcal{R}^d$

(2a)$f(r,.,.)$ is incessant and $|f(r,x,y)| \leq \mathcal{M}(1 + |x| + |y|)$;

(2b) $\mathcal{E}\int_0^Q |f(q,o,o)|^2 dq < \infty$.

(2c)$|f(r,x_1,y_1) - f(r,x_2,y_2)|^2 \leq P|x_1 - x_2|^2 + |y_1 - y_2|^2, |g_i(r,x_1,y_1) - g_i(r,x_2,y_2)|^2 \leq P_i|x_1 - x_2|^2 + \beta_i|y_1 - y_2|^2$

where $\mathcal{M}, P, P_i$ and β_i are non negative constants with $\sum_{i=1}^{\infty.} P_i < \infty$. and

$$\beta = \sum_{i=1}^{\infty.} \beta_i < 1$$

3. Main result

In order to get the solution of (2), we consider the following RBSDEs driven by the finite Brownian motions:

$X_q = \zeta + \int_q^Q f(r, X_r, Y_r)\, dr + \sum_{i=1}^{\infty} \int_q^Q g_i(r, X_r, Y_r) d\alpha_i(r)$- $\int_q^Q Y_r dW^G(r) + \mathbb{K}_Q - \mathbb{K}_q, 0 \le q \le Q$ …(3)

Firstly, we consider a special case of (3), the functions f and g do not depend on (X,Y)

$X_q = \zeta + \int_q^Q f(r)\, dr + \sum_{i=1}^{n} \int_q^Q g_i(r) d\alpha_i(r)$- $\int_q^Q Y_r dW^G(r) + \mathbb{K}_Q - \mathbb{K}_q, 0 \le q \le Q, n \ge 1$ …(4)

Theorem 2:

Assume that (i) –(ii) ,$f \in \mathcal{H}^2, g \in \mathcal{H}^2$ then,there exists a *triple* $(X_r, Y_r, \mathbb{K}_q)_{0 \le q \le Q} \in \mathcal{H}^2 \times \mathcal{J}^2 \times \mathcal{J}^2$

Proof:

Let $=\mathcal{F}_{q=}\mathcal{K}_q^{\mathbb{W}^G} \vee (\vee_{i=1}^{\infty} \mathcal{K}_{q,Q}^{\alpha_i})$ …(5)

Define $\mu = \{\mu_q\}_{0 \le q \le Q}\ as$

$\mu_q = \zeta 1_{\{0 \le q \le Q\}} + R_q 1_{q<Q} + \int_0^q f(r) dr +$ $\sum_{i=1}^{n} \int_0^q g_i(r)\, d\alpha_i(r)$ …(6)

Then, μ is $\mathcal{F}_q$ adopted continuous process; furthermore;

$\sup_{0 \le q \le Q} |\mu_q| \in l^2(\Lambda)$ …(7)

The snell envelope ofμ is given by

$R_q(\mu) = \operatorname{ess\,sup}_{v \in \mathcal{K}} \mathbb{E}[\mu_v\ such\ that\ \mathcal{F}_q]$ …(8)

Where $\mathcal{K}$ is set of all $\mathcal{F}_q$ stopping time such that $0 \le v \le Q$

By the definition of μ we can deduce that $\mathbb{E}\left[\sup_{0 \le q \le Q} |R_q(\mu)|^2\right] < \infty$ …(9)

Due to the doop-meyer decomposition, we have

$R_q(\mu)$=$\mathbb{E}\big[\zeta + \int_0^q f(r)dr + \sum_{i=1}^{n} \int_0^q g_i(r)\, d\alpha_i(r) +$ $\mathbb{K}_Q : \mathcal{F}_q\big] - \mathbb{K}_q$ …(10)

Where $\{\mathbb{K}_Q\}_{0 \le q \le Q}$ is a adopted, continuous and non decreasing process such that $\mathbb{K}_0 = 0\ and\ \mathbb{E}{\mathbb{K}_Q}^2 < \infty.so\ we$ have

$\mathbb{E} \sup_{0 \le q \le Q} \left|\mathbb{E}[\zeta + \int_0^q f(r)dr + \sum_{i=1}^{n} \int_0^q g_i(r)\, d\alpha_i(r) + \mathbb{K}_Q : \mathcal{F}_q]\right|^2 < \infty$ …(11)

Martingale representation theorem yields that there exists $\mathcal{F}_q$ progressive measurable process $\{Y_q\} \in \mathcal{R}^d$

$$\mathcal{M}_q \triangleq \mathbb{E}\left[\zeta + \int_0^q f(r)dr + \sum_{i=1}^{n} \int_0^q g_i(r)\, d\alpha_i(r) + \mathbb{K}_Q : \mathcal{F}_q\right]$$

$=\mathcal{M}_0 + \int_0^q Y_r dW^G(r), 0 \le q \le Q$ …(12)

let $X_q = esssup_{v \in \mathcal{K}}\ \mathbb{E}\left[\zeta 1_{v=Q} + R_{v<Q} + \int_q^v f(r)dr + \sum_{i=1}^{n} \int_q^v g_i(r) d\alpha_i(r) | \mathcal{F}_q\right]; then,$

$$X_q + \int_0^q f(r)dr + \sum_{i=1}^{n} \int_q^v g_i(r) d\alpha_i(r)$$

$=R_q(\mu) = \mathcal{M}_q - \mathbb{K}_q$ …(13)

$=\mathcal{M}_0 + \int_0^q y_r dw^G(r) - \mathbb{K}_q$

therefore, $X_q = \zeta + \int_q^Q f(r)\, dr + \sum_{i=1}^{n} \int_q^Q g_i(r) d\alpha_i(r)$- $\int_q^Q Y_r dW^G(r) + \mathbb{K}_Q - \mathbb{K}_q$…(14)

by the definition of $X_q$and $R_q(\mu), \zeta \ge R_Q$,

$$X_q + \int_0^q f(r)dr + \sum_{i=1}^{n} \int_0^q g_i(r)\, d\alpha_i(r)$$

$=R_q(\mu), \ge \mu_q$

$\zeta 1_{v=Q} + R_q 1_{q<Q} + \int_0^q f(r)dr + \sum_{i=1}^{n} \int_0^q g_i(r)\, d\alpha_i(r).$

so we have $X_q \le R_q$

finally ,from Hamadene[15],we get $\int_0^Q \left(R_q(\mu) - \mu_q\right) d\mathbb{K}_q = 0$ then

$\int_0^Q (X_q - R_q)\ d\mathbb{K}_q = 0$ …(16)

it shows that the process,$(X_q, Y_q, \mathbb{K}_q)_{0 \le q \le Q}$ is the solution of (4).

Theorem 3:

Under the assumptions of (a)-(c), their exists a unique solution $(X_q, Y_q, \mathbb{K}_q)_{0 \le q \le Q}$ of (3)

proof:

let P $= \mathcal{J}^2 \times \mathcal{H}^2$ be endowed with the norm

$\|X, Y\|_\alpha = \left(\mathbb{E}\left[\int_0^Q e^{\alpha r}(|X_r|^2 + |Y_r|^2)\right)\right)^{1/2}$ …(17)

for a suitable constant $\alpha > 0$ .we define the map φ from P into itself and $(\acute{X}, \acute{Y})$ and $(\acute{X}', \acute{Y}')$ are two elements of P. define (X,Y)= $\varphi(\acute{X}, \acute{Y}), (X', Y') = \varphi(\acute{X}', \acute{Y}')$ where $(X, Y, \mathbb{K})$ and $((X', Y', \mathbb{K}'))$ are the solutions of (4) associated with $(\zeta, f(q, \acute{X}, \acute{Y})), g_i(q, \acute{X}, \acute{Y}), R)$ and $(\zeta, f(q, \acute{X}', \acute{Y}')), g_i(q, \acute{X}', \acute{Y}'), R'))$, respectively . set $(\bar{X}, \bar{Y}) = (X_q - X'_q, Y_q - Y'_q)$ and

$\psi_{\mathcal{M}}(a) = a^2 1_{\{-\mathcal{M} \le a \le \mathcal{M}\}} + \mathcal{M}(2a - \mathcal{M}) 1_{\{a > M\}} - \mathcal{M}(2a + \mathcal{M}) 1_{a < -M}$ …(18)

if we define $\psi'_{\mathcal{M}}(a)/a = 2$ when a=0,then $0 \leq \psi'_{\mathcal{M}}(\bar{X}_r)/\bar{X}_r \leq 2$. applying ito formula to $e^{\alpha q}\psi_{\mathcal{M}}(\bar{X}_r)$, we have $e^{\alpha q}\psi_{\mathcal{M}}(\bar{X}_r) +$
$\alpha \int_q^Q e^{\alpha r}\psi_{\mathcal{M}}(\bar{X}_r)dr + \int_q^Q e^{\alpha r}1_{\{-\mathcal{M} \leq \bar{X}_r \leq \mathcal{M}\}}|\bar{Y}_r|^2 dr$
$=\int_q^Q e^{\alpha q}\psi_{\mathcal{M}}(\bar{X}_r)(f(r,\bar{X}_r,\bar{Y}_r) - (f(r,\bar{X}'_r,\overline{Y'_r})dr +$
$\sum_{i=1}^n \int_q^Q e^{\alpha q}1_{\{-\mathcal{M} \leq \bar{X}_r \leq \mathcal{M}\}}\left|g_i(r,\bar{X}_r,\bar{Y}_r) - g_i\left(r,\bar{X}'_r,\overline{Y'_r}\right)\right|^2 dr -$
$\sum_{i=1}^n \int_q^Q e^{\alpha q}\psi'_{\mathcal{M}}(\bar{X}_r)\left(g_i(r,\bar{X}_r,\bar{Y}_r) - g_i\left(r,\bar{X}'_r,\overline{Y'_r}\right)\right)d\alpha_i(r)$
$-\int_q^Q e^{\alpha q}\psi'_{\mathcal{M}}(\bar{X}_r)\bar{Y}_r dw^G(r) + e^{\alpha q}\psi'_{\mathcal{M}}(\bar{X}_r)(d\mathbb{K}_r - d\mathbb{K}'_r)$. …(19)

taking expectation on both sides of (19) and noticing that $\int_q^Q e^{\alpha q}\psi'_{\mathcal{M}}(\bar{X}_r)(d\mathbb{K}_r - d\mathbb{K}'_r) \leq 0$,we have

$$\mathbb{E}e^{\alpha q}\psi_{\mathcal{M}}(\bar{X}_r) + \mathbb{E}\alpha\int_q^Q e^{\alpha r}\psi_{\mathcal{M}}(\bar{X}_r)dr + \mathbb{E}\int_q^Q e^{\alpha r}1_{\{-\mathcal{M} \leq \bar{X}_r \leq \mathcal{M}\}}|\bar{Y}_r|^2 dr$$
$$\leq \mathbb{E}\int_q^Q e^{\alpha r}\psi'_{\mathcal{M}}(\bar{X}_r)(f(r,\bar{X}_r,\bar{Y}_r) - (f(r,\bar{X}'_r,\overline{Y'_r})dr + \sum_{i=1}^n \int_q^Q e^{\alpha r}1_{\{-\mathcal{M} \leq \overline{Y_r} \leq \mathcal{M}\}}\left|g_i(r,\bar{X}_r,\bar{Y}_r) - g_i\left(r,\bar{X}'_{r,}\overline{Y'_r}\right)\right|^2 dr$$
$$\leq 2\mathbb{E}\int_q^Q e^{\alpha q}\,\bar{Y}_r(f(r,\bar{X}_r,\bar{Y}_r) - (f(r,\bar{X}'_r,\overline{Y'_r})dr + \sum_{i=1}^n \mathbb{E}\int_q^Q e^{\alpha r}\left|g_i(r,\bar{X}_r,\bar{Y}_r) - g_i\left(r,\bar{X}'_{r,}\overline{Y'_r}\right)\right|^2 dr$$
$$\leq \frac{2P}{1-\alpha}\mathbb{E}\int_q^Q e^{\alpha r}|\overline{X_r}|^2 + \left(\sum_{i=1}^\infty P_i + \frac{1-\alpha}{2}\right)\mathbb{E}\int_q^Q e^{\alpha r}\left|\overline{X_r} - \overline{X_r'}\right|^2 dr + \frac{1+\alpha}{2}\mathbb{E}\int_q^Q e^{\alpha r}\left|\overline{Y_r} - \overline{Y'_r}\right|^2 dr \quad …(20)$$

let $\lambda = \frac{2P}{1} - \alpha, \bar{P} = 2(\sum_{i=1}^\infty P_i + (1-\alpha)/2))/(1+\alpha)$, $\delta = \lambda + P_i$,and $\mathcal{M} \to \infty$; we have

$$\bar{P}\mathbb{E}\int_q^Q e^{\alpha r}\left|\overline{X_r} - \overline{X_r'}\right|^2 dr + \mathbb{E}\int_q^Q e^{\alpha r}|\overline{Y_r} - \overline{Y'_r}|^2 dr$$
$$\leq \frac{1+\alpha}{2}\mathbb{E}\int_q^Q e^{\alpha r}\overline{(P}\left|\overline{X_r} - \overline{X_r'}\right|^2 + |\overline{Y_r} - \overline{Y'_r}|^2) \quad …(21)$$

that is $\|X_r, Y_r\|_\alpha^2 \leq \frac{1+\alpha}{2}\|X'_r, Y'_r\|_\alpha^2$ …(22)

It follows that λ is astrict contradiction on with the norm $\|.\|_\alpha$ where α is defined as above. Then λ has a fixed point $(X, Y, \mathbb{K})$ which is the unique solution of (4) from the Burholder-Davis–Gundy inequality.

Theorem 4:

Under the conditions of (a)-(c) , their exists a unique solution $(X_q, Y_q, \mathbb{K}_q)_{\ 0 \leq q \leq Q} \in$

Proof(existence)by the theorem 3 ,for any n≥ 1, there exists a unique solution of (3),denoted by $((X_q{}^n, Y_q{}^n, \mathbb{K}_q{}^n))$,

$Y_q{}^n = \zeta\int_q^Q f(r, X_r{}^n, Y_r{}^n)dr + \sum_{i=1}^n \int_q^Q g_i(r, X_r{}^n, Y_r{}^n)d\alpha_i(r)$-$\int_q^Q Y_r^n dW^G(r) + \mathbb{K}_Q{}^n - \mathbb{K}_q{}^n s$ …(23)

In the following parts , we will claim that $(X_q{}^n, Y_q{}^n, \mathbb{K}_q{}^n)$ is a Cauchy sequence in $\mathcal{J}^2 \times \mathcal{J}^2 \times \mathcal{H}^2$ .without loss of generality. we let $n < m$.applying general formula to $\left|X_q{}^n - X_q{}^m\right|^2$.we have

$$\left|X_q{}^n - X_q{}^m\right|^2 + \int_q^Q |Y_r{}^n - Y_r{}^m|^2 dr$$
$$=2\int_q^Q (X_r{}^n - X_r{}^m)(f(r, X_r{}^n, Y_r{}^n) - f(r, X_r{}^m, Y_r{}^m)dr + \sum_{i=n+1}^m \int_q^Q |g_i(r, X_r{}^n, Y_r{}^n) - g_i(r, X_r{}^m, Y_r{}^m)|^2 dr$$
$$- 2\sum_{i=n+1}^m \int_q^Q (X_r{}^n - X_r{}^m)(g_i(r, X_r{}^n, Y_r{}^n) - g_i(r, X_r{}^m, Y_r{}^m)d\alpha_i(r)$$
$$-2\int_q^Q (X_r{}^n - X_r{}^m)(Y_r{}^n - Y_r{}^m)dW^G(r) + 2\int_q^Q (X_r{}^n - X_r{}^m)(d\mathbb{K}_r{}^n - \mathbb{K}_r{}^m) \quad …(24)$$

Taking expectation on both sides of (24) and nothing that $\int_q^Q (X_r{}^n - X_r{}^m)(d\mathbb{K}_r{}^n - \mathbb{K}_r{}^m) \leq 0$, we obtain

$$E\left|X_q{}^n - X_q{}^m\right|^2 + E|Y_r{}^n - Y_r{}^m|^2 dr$$
$$\leq 2E\int_q^Q (X_r{}^n - X_r{}^m)\left(f(r, X_r{}^n, Y_r{}^n) - f(r, X_r{}^m, Y_r{}^m)dr\right.$$
$$+\sum_{i=n+1}^m E\int_q^Q |g_i(r, X_r{}^n, Y_r{}^n) - g_i(r, X_r{}^m, Y_r{}^m)|^2 dr \quad …(25)$$

By (c) and elementary inequality $2xy \leq \alpha x^2 + \left(\frac{1}{\alpha}\right)y^2, \alpha > 0$, we obtain

$$E\left|X_q{}^n - X_q{}^m\right|^2 + E\int_q^Q |Y_r{}^n - Y_r{}^m|^2 dr$$
$$\leq \frac{2P}{1-\beta}E\int_q^Q |X_r{}^n - X_r{}^m|^2 dr + (1-\beta)/2\,E\int_q^Q |X_r{}^n - X_r{}^m|^2 dr$$
$$+(1-\beta)/2\,E\int_q^Q |Y_r{}^n - Y_r{}^m|^2 dr + \beta E\int_q^Q |Y_r{}^n - Y_r{}^m|^2 dr$$
$$+[\sum_{i=n+1}^m P_i]E\int_q^Q |X_r{}^n - X_r{}^m|^2 dr\, |X_r{}^n - X_r{}^m|^2 \quad …(26)$$

Furthermore,

$E\left|X_q{}^n - X_q{}^m\right|^2 + 1 - \beta/2 \qquad E\int_q^Q |Y_r{}^n - Y_r{}^m|^2 dr \leq$

$P_c\ E\left|X_q{}^n - X_q{}^m\right|^2 dr$ …(27)

Where $P_c = \left(\frac{2P}{1-\beta}\right) + \frac{1-\beta}{2} + \sum_{i=n+1}^{m} P_i$

By Gronwell's inequality and Burholder-Davis-Gundy inequality $E\left[\sup_{0\le q\le Q}\int_q^Q |Y_r{}^n - Y_r{}^m|^2 dr\right] \to 0$ …(28)

Denote the limit of $(X_q{}^n, Y_q{}^n, \mathbb{K}_q{}^n)$ by $(X_q, Y_q, \mathbb{K}_q)$; we will show that $(X_q, Y_q, \mathbb{K}_q)$ satisfies (2). If it is necessary ,we can choose a subsequence of (3). By Holder's inequality,

$$E\left|\int_q^Q f(r, X_r, Y_r) - f((r, X_r{}^n, Y_r{}^n))\right|^2$$

$$\le QE\left|\int_q^Q f(r, X_r, Y_r) - f((r, X_r{}^n, Y_r{}^n))\right|^2 dr \to 0 \ \ldots(29)$$

From (27),we know $E\int_0^Q \left|X_q{}^n - X_q{}^m\right|^2 \to 0$ …(30)

and $X_q{}^n \to X_q$, so $\sqrt{E\int_0^Q \left|X_q{}^{n+1} - X_q{}^m\right|^2} \le \frac{1}{2^n}$ …(31) for any n,

$\left|X_q{}^n\right| \le \left|X_q{}^1\right| + \sum_{k=1}^{n-1}\left|X_q{}^{k+1} - X_q{}^k\right| \le \left|X_q{}^1\right| + \sum_{k=1}^{\infty}\left|X_q{}^{k+1} - X_q{}^k\right|$ …(32)

Then ,we have $\sqrt{E\int_0^Q \sup_n \left|X_q{}^n\right|^2}$dq

$$\le \sqrt{E\int_0^Q \left(\left|X_q{}^1\right| + \sum_{k=1}^{\infty}\left|X_q{}^{k+1} - X_q{}^k\right|\right)^2 dq}$$

$$\le \sqrt{E\int_0^Q \left|X_q{}^1\right|^2 dq} + \sum_{k=1}^{\infty}\sqrt{E\int_0^Q \left|X_q{}^{k+1} - X_q{}^k\right|^2 \left|X_q{}^{k+1} - X_q{}^k\right|^2 dq} \ \ldots(33)$$

$$\le \sqrt{E\int_0^Q \left|X_q{}^1\right|^2 dq} + \sum_{k=1}^{\infty}\frac{1}{2^k}$$

$$E\int_0^Q \sup_n |f(r, X_r, Y_r) - f((r, X_r{}^n, Y_r{}^n)|^2 dr$$

$$\le 2PE\int_0^Q \sup_n |X_r{}^n|^2 + |X_r|^2 + \sup_n |Y_r{}^n|^2 + |Y_r|^2\, dr < \infty$$

…(34)

Applying Lebesgue convergence theorem, we deduce that $(X_q, Y_q, \mathbb{K}_q)$ is the solution of (2) by the continuity of the functions f and g.

Uniqueness: Let $\left(X_q{}^k, Y_q{}^k, \mathbb{K}_q{}^k\right)$ $(k = 1,2)$ be two solutions of (2), $\overline{X_q} = X_q{}^1 - X_q{}^2$, $\overline{Y_q} = Y_q{}^1 - Y_q{}^2$. We apply Ito formula to $e^{\alpha q}\psi_{\mathcal{M}}(\overline{X_q}) + \alpha\int_q^Q e^{\alpha r}\psi_{\mathcal{M}}(\overline{Y_r})dr + \int_q^Q e^{\alpha r} 1_{\{-\mathcal{M}\le\overline{Y_r}\le\mathcal{M}\}}\overline{Y_q}^2 dr$

$= \int_q^Q e^{\alpha r}\psi'_{\mathcal{M}}(\overline{X_r})(f(r, X_r{}^1, Y_r{}^1) - f(r, X_r{}^2, Y_r{}^2)dr + \sum_{k=1}^{\infty}\int_q^Q e^{\alpha r} 1_{\{-\mathcal{M}\le\overline{Y_r}\le\mathcal{M}\}}\left|g_i((r, X_r{}^1, Y_r{}^1) - g_i((r, X_r{}^2, Y_r{}^2))\right|^2 dr - \sum_{k=1}^{\infty}\int_q^Q e^{\alpha r}\psi'_{\mathcal{M}}(\overline{X_r}) \times (g_i((r, X_r{}^1, Y_r{}^1) - g_i((r, X_r{}^2, Y_r{}^2))d\alpha_i(r)$

$-\int_q^Q e^{\alpha r}\psi'_{\mathcal{M}}(\overline{X_r})\,\overline{Y_q}\,dw_r^G + \int_q^Q e^{\alpha r}\psi'_{\mathcal{M}}(\overline{X_r})(d\mathbb{K}_r^1 - d\mathbb{K}_r^2)$….(35)

Taking expectation of (35),

$Ee^{\alpha q}\psi_{\mathcal{M}}(\overline{X_q}) + \alpha E\int_q^Q e^{\alpha r}\psi_{\mathcal{M}}(\overline{Y_r})dr + E\int_q^Q e^{\alpha r} 1_{\{-\mathcal{M}\le\overline{Y_r}\le\mathcal{M}\}}\overline{Y_q}^2 dr$

$$\le 2E\int_q^Q e^{\alpha r}\psi'_{\mathcal{M}}(\overline{X_r})(f(r, X_r{}^1, Y_r{}^1) - f(r, X_r{}^2, Y_r{}^2)dr$$

$+\sum_{k=1}^{\infty}E\int_q^Q e^{\alpha r} 1_{\{-\mathcal{M}\le\overline{Y_r}\le\mathcal{M}\}}\left|g_i((r, X_r{}^1, Y_r{}^1) - g_i((r, X_r{}^2, Y_r{}^2))\right|^2 dr - \sum_{k=1}^{\infty}\int_q^Q e^{\alpha r}\psi'_{\mathcal{M}}(\overline{X_r})\,dr$

$$\le \left(\frac{2P}{1-\sum_{k=1}^{\infty}\beta_i} + \sum_{k=1}^{\infty}P_i + \frac{1-\sum_{k=1}^{\infty}\beta_i}{2}\right)E\int_q^Q e^{\alpha r}|\overline{X_r}|^2 dr$$

$$+\frac{1+\sum_{k=1}^{\infty}\beta_i}{2}E\int_q^Q e^{\alpha r}\left|\overline{Y_q}\right|^2 dr$$

Let $\mathcal{M} \to \infty$ and applying monotone convergence theorem,we have

$Ee^{\alpha q}\left|\overline{X_q}\right|^2 + \left(\alpha - \frac{2P}{1-\beta} - \sum_{k=1}^{\infty}P_i - \frac{1-\beta}{2}\right) \times E\int_q^Q e^{\alpha r}|\overline{X_r}|^2 dr + \frac{1-\beta}{2}E\int_q^Q e^{\alpha r}|\overline{Y_r}|^2 dr \le 0$…(37)

When α is taken sufficiently large ,we have $\overline{Y_q}$=0 .

It follows that λ is astrict contradiction on with the norm $\|.\|_\alpha$ where α is defined as above. Then λ has a fixed point $(X, Y, \mathbb{K})$ which is the unique solution of (4) from the Burholder-Davis–Gundy inequality.

Theorem 4:

Under the conditions of (a)-(c) , their exists a unique solution $(X_q, Y_q, \mathbb{K}_q)_{\ 0\le q\le Q} \in$

Proof(existence)by the theorem 3 ,for any n≥ 1, there exists a unique solution of (3),denoted by $((X_q{}^n, Y_q{}^n, \mathbb{K}_q{}^n))$,

$Y_q{}^n = \zeta + \int_q^Q f(r, X_r{}^n, Y_r{}^n)dr + \sum_{i=1}^{n}\int_q^Q g_i\,(r, X_r{}^n, Y_r{}^n)d\alpha_i(r) - \int_q^Q Y_r^n dW^G(r) + \mathbb{K}_Q{}^n - \mathbb{K}_q{}^n s$ …(23)

In the following parts , we will claim that $(X_q{}^n, Y_q{}^n, \mathbb{K}_q{}^n)$ is a Cauchy sequence in .without loss of generality. we let $n < m$.applying general formula to $\left|X_q{}^n - X_q{}^m\right|^2$.we have

$\left|X_q{}^n - X_q{}^m\right|^2 + \int_q^Q |Y_r{}^n - Y_r{}^m|^2 dr$

$=2\int_q^Q (X_r{}^n - X_r{}^m)(f(r, X_r{}^n, Y_r{}^n) - f(r, X_r{}^m, Y_r{}^m)dr$

$$+\sum_{i=n+1}^{m}\int_q^Q |g_i(r,X_r{}^n,Y_r{}^n)-g_i(r,X_r{}^m,Y_r{}^m)|^2dr$$
$$-2\sum_{i=n+1}^{m}\int_q^Q (X_r{}^n-X_r{}^m)(g_i(r,X_r{}^n,Y_r{}^n)-g_i(r,X_r{}^m,Y_r{}^m)d\alpha_i(r)$$
$-2\int_q^Q (X_r{}^n-X_r{}^m)(Y_r{}^n-Y_r{}^m)dW^G(r)+2\int_q^Q (X_r{}^n-X_r{}^m)(d\mathbb{K}_r{}^n-\mathbb{K}_r{}^m)$ …(24)

Taking expectation on both sides of (24) and nothing that $\int_q^Q (X_r{}^n-X_r{}^m)(d\mathbb{K}_r{}^n-\mathbb{K}_r{}^m)\le 0$, we obtain

$$E|X_q{}^n-X_q{}^m|^2+E|Y_r{}^n-Y_r{}^m|^2dr$$
$$\le 2E\int_q^Q (X_r{}^n-X_r{}^m)\,(f(r,X_r{}^n,Y_r{}^n)-\ f(r,X_r{}^m,Y_r{}^m)dr$$
$$+\sum_{i=n+1}^{m}E\int_q^Q|g_i(r,X_r{}^n,Y_r{}^n)-g_i(r,X_r{}^m,Y_r{}^m)|^2dr$$

By (c) and elementary inequality $2xy\le\alpha x^2+\left(\frac{1}{\alpha}\right)y^2,\alpha>0$, we obtain

$$E|X_q{}^n-X_q{}^m|^2+E\int_q^Q|Y_r{}^n-Y_r{}^m|^2dr$$
$$\le\frac{2P}{1-\beta}E\int_q^Q|X_r{}^n-X_r{}^m|^2dr+(1-\beta)$$
$$/2\ E\int_q^Q|X_r{}^n-X_r{}^m|^2dr$$
$$+(1-\beta)/2\ E\int_q^Q|Y_r{}^n-Y_r{}^m|^2dr+\beta E\int_q^Q|Y_r{}^n-Y_r{}^m|^2dr$$
$$+[\textstyle\sum_{i=n+1}^{m}P_i]E\int_q^Q|X_r{}^n-X_r{}^m|^2dr\,|X_r{}^n-X_r{}^m|^2 \quad …(26)$$

Furthermore,

$E|X_q{}^n-X_q{}^m|^2+1-\beta/2 \qquad E\int_q^Q|Y_r{}^n-Y_r{}^m|^2dr\le P_c\ E|X_q{}^n-X_q{}^m|^2dr$ …(27)

Where $P_c=\left(\frac{2P}{1-\beta}\right)+\frac{1-\beta}{2}+\sum_{i=n+1}^{m}P_i$

By Gronwell's inequality and Burholder-Davis-Gundy inequality $E\left[\sup_{0\le q\le Q}\int_q^Q|Y_r{}^n-Y_r{}^m|^2dr\right]\to 0$ …(28)

Denote the limit of $(X_q{}^n,Y_q{}^n,\mathbb{K}_q{}^n)$ by $(X_q,Y_q,\mathbb{K}_q)$; we will show that $(X_q,Y_q,\mathbb{K}_q)$ satisfies (2). If it is necessary ,we can choose a subsequence of (3). By Holder's inequality,

$$E\left|\int_q^Q f(r,X_r,Y_r)-f((r,X_r{}^n,Y_r{}^n))\right|^2$$
$$\le QE\left|\int_q^Q f(r,X_r,Y_r)-f((r,X_r{}^n,Y_r{}^n))\right|^2dr\to 0 \quad …(29)$$

From (27),we know $E\int_0^Q|X_q{}^n-X_q{}^m|^2\to 0$ …(30) and $X_q{}^n\to X_q$, so $\sqrt{E\int_0^Q|X_q{}^{n+1}-X_q{}^m|^2}\le\frac{1}{2^n}$ …(31) for any n,

$|X_q{}^n|\le|X_q{}^1|+\sum_{k=1}^{n-1}|X_q{}^{k+1}-X_q{}^k|\le|X_q{}^1|+\sum_{k=1}^{\infty}|X_q{}^{k+1}-X_q{}^k|$ …(32)

Then ,we have $\sqrt{E\int_0^Q\sup_n|X_q{}^n|^2}$dq

$$\le\sqrt{E\int_0^Q\left(|X_q{}^1|+\sum_{k=1}^{\infty}|X_q{}^{k+1}-X_q{}^k|\right)^2dq}$$
$$\le\sqrt{E\int_0^Q|X_q{}^1|^2dq}$$
$$+\sum_{k=1}^{\infty}\sqrt{E\int_0^Q|X_q{}^{k+1}-X_q{}^k|^2\,|X_q{}^{k+1}-X_q{}^k|^2dq} \quad …(33)$$
$$\le\sqrt{E\int_0^Q|X_q{}^1|^2dq}+\sum_{k=1}^{\infty}\frac{1}{2^k}$$
$$E\int_0^Q\sup_n|f(r,X_r,Y_r)-f((r,X_r{}^n,Y_r{}^n)|^2dr$$
$$\le 2PE\int_0^Q\sup_n|X_r{}^n|^2+|X_r|^2+\sup_n|Y_r{}^n|^2+|Y_r|^2\,dr<\infty$$
…(34)

Applying Lebesgue convergence theorem, we deduce that $(X_q,Y_q,\mathbb{K}_q)$ is the solution of (2) by the continuity of the functions f and g.

Uniqueness: Let $(X_q{}^k,Y_q{}^k,\mathbb{K}_q{}^k)$ $(k=1,2)$ be two solutions of (2), $\overline{X_q}=X_q{}^1-X_q{}^2$, $\overline{Y_q}=Y_q{}^1-Y_q{}^2$. We apply Ito formula to

$$e^{\alpha q}\psi_{\mathcal{M}}(\overline{X_q})+\alpha\int_q^Q e^{\alpha r}\psi_{\mathcal{M}}(\overline{Y_r})dr+\int_q^Q e^{\alpha r}1_{\{-\mathcal{M}\le\overline{Y_r}\le\mathcal{M}\}}\overline{Y_q}^2dr$$

$=\int_q^Q e^{\alpha r}\psi'_{\mathcal{M}}(\overline{X_r})(f(r,X_r{}^1,Y_r{}^1)-f(r,X_r{}^2,Y_r{}^2)dr+\sum_{k=1}^{\infty}\int_q^Q e^{\alpha r}1_{\{-\mathcal{M}\le\overline{Y_r}\le\mathcal{M}\}}|g_i((r,X_r{}^1,Y_r{}^1)-g_i((r,X_r{}^2,Y_r{}^2))|^2dr-\sum_{k=1}^{\infty}\int_q^Q e^{\alpha r}\psi'_{\mathcal{M}}(\overline{X_r})\times(g_i((r,X_r{}^1,Y_r{}^1)-g_i((r,X_r{}^2,Y_r{}^2))d\alpha_i(r)$
$-\int_q^Q e^{\alpha r}\psi'_{\mathcal{M}}(\overline{X_r})\,\overline{Y_q}dw_r^G+\int_q^Q e^{\alpha r}\psi'_{\mathcal{M}}(\overline{X_r})(d\mathbb{K}_r^1-d\mathbb{K}_r^2)$….(35)

Taking expectation of (35),

$Ee^{\alpha q}\psi_{\mathcal{M}}(\overline{X_q})+\alpha E\int_q^Q e^{\alpha r}\psi_{\mathcal{M}}(\overline{Y_r})dr+E\int_q^Q e^{\alpha r}1_{\{-\mathcal{M}\le\overline{Y_r}\le\mathcal{M}\}}\overline{Y_q}^2dr$

$$\le 2E\int_q^Q e^{\alpha r}\psi'_{\mathcal{M}}(\overline{X_r})(f(r,X_r{}^1,Y_r{}^1)-f(r,X_r{}^2,Y_r{}^2)dr$$
$$+\sum_{k=1}^{\infty}E\int_q^Q e^{\alpha r}1_{\{-\mathcal{M}\le\overline{Y_r}\le\mathcal{M}\}}|g_i((r,X_r{}^1,Y_r{}^1)-$$

$g_i((r, X_r{}^2, Y_r{}^2))\big|^2 dr - \sum_{k=1}^{\infty} \int_q^Q e^{\alpha r} \psi'_{\mathcal{M}}(\overline{X_r})\, dr$

$$\leq \left(\frac{2P}{1 - \sum_{k=1}^{\infty} \beta_i} + \sum_{k=1}^{\infty} P_i + \frac{1 - \sum_{k=1}^{\infty} \beta_i}{2} \right) E \int_q^Q e^{\alpha r} |\overline{X_r}|^2 dr$$

$$+ \frac{1 + \sum_{k=1}^{\infty} \beta_i}{2} E \int_q^Q e^{\alpha r} \left|\overline{Y_q}\right|^2 dr \ldots (36s)$$

Let $\mathcal{M} \to \infty$ and applying monotone convergence theorem,we have

$$E e^{\alpha q} \left|\overline{X_q}\right|^2 + \left(\alpha - \frac{2P}{1-\beta} - \sum_{k=1}^{\infty} P_i - \frac{1-\beta}{2} \right) \times$$

$$E \int_q^Q e^{\alpha r} |\overline{X_r}|^2 dr + \frac{1-\beta}{2} E \int_q^Q e^{\alpha r} |\overline{Y_r}|^2 dr \leq 0 \ldots (37)$$

When α is taken sufficiently large ,we have $\overline{Y_q}$=0 .

4. Conclusion

This paper has some concepts of reflected backward stochastic differential equations driven by countable G-Brownian motions. The existence and uniqueness of RBSDEs are obtained.

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