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Application of Linear Programming in Mathematics and Approach for Optimal Solution

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***Abstract*: Linear programming, mathematical modeling technique in which a linear function is maximized or minimized when subjected to various constraints. This technique has been useful for guiding quantitative decisions in business planning, in industrial engineering, and—to a lesser extent—in the social and physical sciences. Optimization is the way of life. We all have finite resources and time and we want to make the most of them. From using your time productively to solving supply chain problems for your company – everything uses optimization. It's a especially interesting and relevant topic in data science. It is also a very interesting topic – it starts with simple problems, but can get very complex. For example, sharing a chocolate between siblings is a simple optimization problem. We don't think in mathematical term while solving it. On the other hand devising inventory and warehousing strategy for an e-tailer can be very complex. Millions of SKUs with different popularity in different regions to be delivered in defined time and resources – you see what I mean! Linear programming (LP) is one of the simplest ways to perform optimization. It helps you solve some very complex optimization problems by making a few simplifying assumptions. As an analyst you are bound to come across applications and problems to be solved by Linear Programming. For some reason, LP doesn't get as much attention as it deserves while learning data science. So, I thought let me do justice to this awesome technique. I decided to write an article which explains Linear programming in simple English. I have kept the content as simple as possible. The idea is to get you started and excited about Linear Programming.**

***Keywords*: Linear programming, Standard form, Augmented form, Duality, Special cases, Complementary slackness**

1. Introduction

In mathematics, Linear programming (LP) is a technique for optimization of a linear objective function, subject to linear equality and linear inequality constraints. Informally, linear programming determines the way to achieve the best outcome (such as maximum profit or lowest cost) in a given mathematical model and given some list of requirements represented as linear equations. More formally, given a polytope (for example, a polygon or a polyhedron), and a real-valued affine function

$$f(x_1, x_2, \dots, x_n) = c_1x_1 + c_2x_2 + \cdots + c_nx_n + d$$

defined on this polytope, a linear programming method will find a point in the polytope where this function has the smallest (or largest) value. Such points may not exist, but if they do, searching through the polytope vertices is guaranteed to find at least one of them. Linear programs are problems that can be expressed in canonical form:

Maximize $\mathbf{c}^T\mathbf{x}$
Subject to $A\mathbf{x} \le \mathbf{b}$.

X represents the vector of variables (to be determined), while c and b are vectors of (known) coefficients and A is a (known) matrix of coefficients. The expression to be maximized or minimized is called the objective function ($\mathbf{c}^T\mathbf{x}$ in this case). The equations $A\mathbf{x} \le \mathbf{b}$ are the constraints which specify a convex polyhedron over which the objective function is to be optimized. Linear programming can be applied to various fields of study. Most extensively it is used in business and economic situations, but can also be utilized for some engineering problems. Some industries that use linear programming models include transportation, energy, telecommunications, and manufacturing. It has proved useful in modeling diverse types of problems in planning, routing, scheduling, assignment, and design. Geometrically, the linear constraints define a convex polyhedron, which is called the feasible region. Since the objective function is also linear, hence a convex function, all local optima are automatically global optima (by the KKT theorem). The linearity of the objective function also implies that the set of optimal solutions is the convex hull of a finite set of points - usually a single point. There are two situations in which no optimal solution can be found. First, if the constraints contradict each other (for instance, $x \ge 2$ and $x \le 1$) then the feasible region is empty and there can be no optimal solution, since there are no solutions at all. In this case, the LP is said to be infeasible. Alternatively, the polyhedron can be unbounded in the direction of the objective function (for example: maximize $x_1 + 3\,x_2$ subject to $x_1 \ge 0$, $x_2 \ge 0$, $x_1 + x_2 \ge 10$), in which case there is no optimal solution since solutions with arbitrarily high values of the objective function can be constructed. Barring these two pathological conditions (which are often ruled out by resource constraints integral to the problem being represented, as above), the optimum is always attained at a vertex of the polyhedron. However, the optimum is not necessarily unique: it is possible to have a set of optimal solutions covering an edge or face of the polyhedron, or even the entire polyhedron (This last situation would occur if the objective function were constant).

A. What is Linear Programming?

Now, what is linear programming? Linear programming is a simple technique where we depict complex relationships through linear functions and then find the optimum points. The important word in previous sentence is depict. The real relationships might be much more complex – but we can simplify them to linear relationships. Applications of linear programming are everywhere around you. You use linear programming at personal and professional fronts. You are using linear programming when you are driving from home to work and want to take the shortest route. Or when you have a project delivery you make strategies to make your team work efficiently for on time delivery.

Example of a linear programming problem

Let's say a FedEx delivery man has 6 packages to deliver in a day. The warehouse is located at point A. The 6 delivery destinations are given by U, V, W, X, Y and Z. The numbers on the lines indicate the distance between the cities. To save on fuel and time the delivery person wants to take the shortest route.

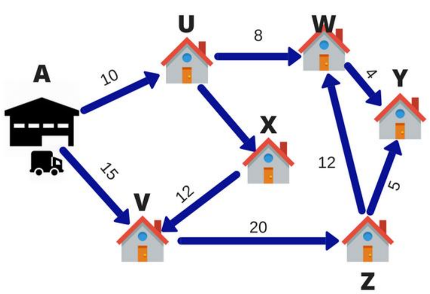

Fig. 1. Linear programming problem

So, the delivery person will calculate different routes for going to all the 6 destinations and then come up with the shortest route. This technique of choosing the shortest route is called linear programming. In this case, the objective of the delivery person is to deliver the parcel on time at all 6 destinations. The process of choosing the best route is called Operation Research. Operation research is an approach to decision-making, which involves a set of methods to operate a system. In the above example, my system was the Delivery model. Linear programming is used for obtaining the most optimal solution for a problem with given constraints. In linear programming, we formulate our real life problem into a mathematical model. It involves an objective function, linear inequalities with subject to constraints. Is the linear representation of the 6 points above representative of real world? Yes and No. It is oversimplification as the real route would not be a straight line. It would likely have multiple turns, U turns, signals and traffic jams. But with a simple assumption, we have reduced the complexity of the problem drastically and are creating a solution which should work in most scenarios.

B. Objectives

After studying this unit, one should be able to:

- Understand the basic concept of Linear Programming
- Analyze the uses of Linear Programming
- Know the different concepts of standard form and augmented form problems.
- Have the knowledge of complementary slackness theorem

2. Uses of linear programming

Linear programming is a considerable field of optimization for several reasons. Many practical problems in operations research can be expressed as linear programming problems. Certain special cases of linear programming, such as network flow problems and multi commodity flow problems are considered important enough to have generated much research on specialized algorithms for their solution. A number of algorithms for other types of optimization problems work by solving LP problems as sub-problems. Historically, ideas from linear programming have inspired many of the central concepts of optimization theory, such as duality, decomposition, and the importance of convexity and its generalizations. Likewise, linear programming is heavily used in microeconomics and company management, such as planning, production, transportation, technology and other issues. Although the modern management issues are ever-changing, most companies would like to maximize profits or minimize costs with limited resources. Therefore, many issues can boil down to linear programming problems.

3. Standard form

Standard form is the usual and most intuitive form of describing a linear programming problem. It consists of the following three parts:

- A linear function to be maximized

e.g. maximize $c_1x_1 + c_2x_2$

- Problem constraints of the following form

e.g

$$a_{11}x_1 + a_{12}x_2 \leq b_1$$
$$a_{21}x_1 + a_{22}x_2 \leq b_2$$
$$a_{31}x_1 + a_{32}x_2 \leq b_3$$

- Non-negative variables

e.g. $x_1 \geq 0$

$x_2 \geq 0$

The problem is usually expressed in matrix form, and then becomes:

maximize $\mathbf{c}^{\mathrm{T}}\mathbf{x}$
subject to $A\mathbf{x} \leq \mathbf{b}, \; \mathbf{x} \geq 0$

Other forms, such as minimization problems, problems with constraints on alternative forms, as well as problems involving negative variables can always be rewritten into an equivalent problem in standard form.

Example 1

Suppose that a farmer has a piece of farm land, say A square kilometres large, to be planted with either wheat or barley or some combination of the two. The farmer has a limited permissible amount F of fertilizer and P of insecticide which can be used, each of which is required in different amounts per unit area for wheat (F_1, P_1) and barley (F_2, P_2). Let S_1 be the selling price of wheat, and S_2 the price of barley. If we denote the area planted with wheat and barley by x_1 and x_2 respectively, then the optimal number of square kilometres to plant with wheat vs barley can be expressed as a linear programming problem:

maximize	$S_1x_1 + S_2x_2$	(maximize the revenue — revenue is the "objective function")
subject to	$x_1 + x_2 \leq A$	(limit on total area)
	$F_1x_1 + F_2x_2 \leq F$	(limit on fertilizer)
	$P_1x_1 + P_2x_2 \leq P$	(limit on insecticide)
	$x_1 \geq 0, \; x_2 \geq 0$	(cannot plant a negative area)

Which in matrix form becomes:

maximize $\begin{bmatrix} S_1 & S_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

subject to $\begin{bmatrix} 1 & 1 \\ F_1 & F_2 \\ P_1 & P_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} A \\ F \\ P \end{bmatrix}, \; \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \geq 0$

4. Augmented form (slack form)

Linear programming problems must be converted into augmented form before being solved by the simplex algorithm. This form introduces non-negative slack variables to replace inequalities with equalities in the constraints. The problem can then be written in the following form:

Maximize Z in:

$$\begin{bmatrix} 1 & -\mathbf{c}^T & 0 \\ 0 & \mathbf{A} & \mathbf{I} \end{bmatrix} \begin{bmatrix} Z \\ \mathbf{x} \\ \mathbf{x}_s \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{b} \end{bmatrix}$$

$$\mathbf{x}, \; \mathbf{x}_s \geq 0$$

where $\mathbf{x}_s$ are the newly introduced slack variables, and Z is the variable to be maximized.

Example 2

The example above becomes as follows when converted into augmented form:

maximize	$S_1x_1 + S_2x_2$	(objective function)
subject to	$x_1 + x_2 + x_3 = A$	(augmented constraint)
	$F_1x_1 + F_2x_2 + x_4 = F$	(augmented constraint)
	$P_1x_1 + P_2x_2 + x_5 = P$	(augmented constraint)
	$x_1, x_2, x_3, x_4, x_5 \geq 0$	

where x_3, x_4, x_5 are (non-negative) slack variables.

Which in matrix form becomes:

Maximize Z in:

$$\begin{bmatrix} 1 & -S_1 & -S_2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & F_1 & F_2 & 0 & 1 & 0 \\ 0 & P_1 & P_2 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} Z \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ A \\ F \\ P \end{bmatrix}, \; \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \geq 0$$

5. Duality

Every linear programming problem, referred to as a primal problem, can be converted into a dual problem, which provides an upper bound to the optimal value of the primal problem. In matrix form, we can express the primal problem as:

maximize $\mathbf{c}^T\mathbf{x}$
subject to $\mathbf{A}\mathbf{x} \leq \mathbf{b}, \; \mathbf{x} \geq 0$

The corresponding dual problem is:

minimize $\mathbf{b}^T\mathbf{y}$
subject to $\mathbf{A}^T\mathbf{y} \geq \mathbf{c}, \; \mathbf{y} \geq 0$

where y is used instead of x as variable vector.

There are two ideas fundamental to duality theory. One is the fact that the dual of a dual linear program is the original primal linear program. Additionally, every feasible solution for a linear program gives a bound on the optimal value of the objective function of its dual. The weak duality theorem states that the objective function value of the dual at any feasible solution is always greater than or equal to the objective function value of the primal at any feasible solution. The strong duality theorem states that if the primal has an optimal solution, x*, then the dual also has an optimal solution, y*, such that $c^Tx^*=b^Ty^*$. A linear program can also be unbounded or infeasible. Duality theory tells us that if the primal is unbounded then the dual is infeasible by the weak duality theorem. Likewise, if the dual is unbounded, then the primal must be infeasible. However, it is possible for both the dual and the primal to be infeasible (See also Farkas' lemma).

Example 3

Revisit the above example of the farmer who may grow wheat and barley with the set provision of some A land, F fertilizer and P insecticide. Assume now that unit prices for each of these means of production (inputs) are set by a planning board. The planning board's job is to minimize the total cost of procuring the set amounts of inputs while providing the farmer with a floor on the unit price of each of his crops (outputs), S1 for wheat and S2 for barley. This corresponds to the following linear programming problem:

minimize	$Ay_A + Fy_F + Py_P$	(minimize the total cost of the means of production as the "objective function")
subject to	$y_A + F_1 y_F + P_1 y_P \geq S_1$	(the farmer must receive no less than S_1 for his wheat)
	$y_A + F_2 y_F + P_2 y_P \geq S_2$	(the farmer must receive no less than S_2 for his barley)
	$y_A \geq 0,\ y_F \geq 0,\ y_P \geq 0$	(prices cannot be negative)

Which in matrix form becomes:

minimize $\begin{bmatrix} A & F & P \end{bmatrix} \begin{bmatrix} y_A \\ y_F \\ y_P \end{bmatrix}$

subject to $\begin{bmatrix} 1 & F_1 & P_1 \\ 1 & F_2 & P_2 \end{bmatrix} \begin{bmatrix} y_A \\ y_F \\ y_P \end{bmatrix} \geq \begin{bmatrix} S_1 \\ S_2 \end{bmatrix},\ \begin{bmatrix} y_A \\ y_F \\ y_P \end{bmatrix} \geq 0$

The primal problem deals with physical quantities. With all inputs available in limited quantities, and assuming the unit prices of all outputs is known, what quantities of outputs to produce so as to maximize total revenue? The dual problem deals with economic values. With floor guarantees on all output unit prices, and assuming the available quantity of all inputs is known, what input unit pricing scheme to set so as to minimize total expenditure? To each variable in the primal space corresponds an inequality to satisfy in the dual space, both indexed by output type. To each inequality to satisfy in the primal space corresponds a variable in the dual space, both indexed by input type. The coefficients which bound the inequalities in the primal space are used to compute the objective in the dual space, input quantities in this example. The coefficients used to compute the objective in the primal space bound the inequalities in the dual space, output unit prices in this example. Both the primal and the dual problems make use of the same matrix. In the primal space, this matrix expresses the consumption of physical quantities of inputs necessary to produce set quantities of outputs. In the dual space, it expresses the creation of the economic values associated with the outputs from set input unit prices. Since each inequality can be replaced by an equality and a slack variable, this means each primal variable corresponds to a dual slack variable, and each dual variable corresponds to a primal slack variable. This relation allows us to complementary slackness.

6. Special cases

A packing LP is a linear program of the form,

maximize $\mathbf{c}^{\mathbf{T}}\mathbf{x}$
subject to $\mathbf{A}\mathbf{x} \leq \mathbf{b},\ \mathbf{x} \geq 0$

such that the matrix A and the vectors b and c are non-negative.

The dual of a packing LP is a covering LP, a linear program of the form

minimize $\mathbf{b}^T\mathbf{y}$
subject to $\mathbf{A}^T\mathbf{y} \geq \mathbf{c},\ \mathbf{y} \geq 0$

such that the matrix A and the vectors b and c are non-negative.

Example 4

Covering and packing LPs commonly arise as a linear programming relaxation of a combinatorial problem. For example, the LP relaxation of set packing problem, independent set problem, or matching is a packing LP. The LP relaxation of set cover problem, vertex cover problem, or dominating set problem is a covering LP. Finding a fractional coloring of a graph is another example of a covering LP. In this case, there is one constraint for each vertex of the graph and one variable for each independent set of the graph.

7. Complementary slackness

It is possible to obtain an optimal solution to the dual when only an optimal solution to the primal is known using the complementary slackness theorem. The theorem states:

Suppose that x = $(x_1, x_2, \ldots, x_n)$ is primal feasible and that y = $(y_1, y_2, \ldots, y_m)$ is dual feasible. Let $(w_1, w_2, \ldots, w_m)$ denote the corresponding primal slack variables, and let $(z_1, z_2, \ldots, z_n)$ denote the corresponding dual slack variables. Then x and y are optimal for their respective problems if and only if $x_j z_j = 0$, for $j = 1, 2, \ldots, n$, $w_i y_i = 0$, for $i = 1, 2, \ldots, m$.

So if the ith slack variable of the primal is not zero, then the ith variable of the dual is equal zero. Likewise, if the jth slack variable of the dual is not zero, then the jth variable of the primal is equal to zero.

Activity 1

1. Discuss the uses of Linear Programming.
2. Explain briefly the concept of Duality.

8. Conclusion

Linear programming is an important field of optimization for several reasons. Many practical problems in operations research can be expressed as linear programming problems. Followed by the basic concept the concepts of duality, standard form and augmented form have described in the chapter. The different kind of problems can be solved using Linear Programming approach is discussed in special case section. Further the theorem of Complementary slackness was discussed in brief to have more clear understanding of solution to Linear programming problems.

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