International Journal of Research in Engineering, Science and Management
Volume-2, Issue-2, February-2019
www.ijresm.com | ISSN (Online): 2581-5792

The Prime Number of the Zero Divisor Graph

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***Abstract*: The prime number of zero divisor graph b(Γ(Zn)) shows the maximum value of b(Γ(Zn)), where n is positive integer.**

***Keywords*: prime number, zero divisor graph.**

1. Introduction

In the last three decades graph theory has established itself as a worthwhile mathematical discipline and there are many applications of graph theory to a wide variety of subjects which include operation research, physics, chemistry, economics, genetics, sociology, linguistics, engineering, computer science, etc. The development of many branched in mathematics has been necessitated while considering certain real life problems arising in practical life on problems arising in other science [1]-[2].

2. Mathematic formulation

A. Definition

The prime number of $\Gamma(\mathrm{Z_m})$, denoted by $B\left(\Gamma(\mathrm{Z_m})\right)$ is defined by,

$$\Gamma(\mathrm{Z_m}) = \left\{\frac{|L(S)|}{|S|}, where\ S \subseteq V\left(\Gamma(\mathrm{Z_m})\right), S \neq \emptyset, \quad \mathrm{L(S)} \neq V\left(\Gamma(\mathrm{Z_m})\right)\right\}$$

which satisfies the following conditions;

(i) $\mathrm{L(S)} \cup \mathrm{S} = V\left(\Gamma(\mathrm{Z_m})\right)$
(ii) $L(S) \cap \mathrm{S} \neq \emptyset$
(iii) $d(x) \leq d(y) for\ x \in S\ and\ y \in L(S)$
(iv) No two vertices in S are adjacent.

Lemma 2.2
The domination set $\Gamma(\mathrm{Z_m})$ is connected and m is a composite number.

B. Theorem

The prime number $p > 2$, then $B\left(\Gamma(\mathrm{Z_{2p}})\right) = \frac{1}{p-1}$.

Proof
The set $\Gamma(\mathrm{Z_{2p}})$ is $\{2,4,6,\dots,2(p-1),p\}$. $\Gamma(\mathrm{Z_{2p}})$ be the vertex with a star graph $K_{1,p-1}$. Let S be a non-empty subset of the vertex set $V\left(\Gamma(\mathrm{Z_{2p}})\right)$, then for any $u \in S$, such that $d(x) < d\,(v)$, where $v \in V - S$. Clearly, all the vertices are of minimum degree except p, then $S = \{2,4,\dots.2(p-1)\}$, that is $|S| = p - 1$ and the neighbourhood of the set $S = L(S)$ and $|L(S)| = p - (p-1) = 1$. Hence,

$$B\left(\Gamma(\mathrm{Z_{2p}})\right) = \frac{|L(S)|}{|S|} = \frac{1}{p-1}$$

C. Theorem

For any prime p, $b\left(\Gamma(\mathrm{Z_{p^2}})\right) = \frac{1}{p-2}$.

Proof
The vertex set of $\Gamma(\mathrm{Z_{2p}})$ is $\{p, 2p, 3p, \dots, p(p-1)\}$. Any two vertices in $B\left(\Gamma(\mathrm{Z_{p^2}})\right)$ are adjacent. Clearly,$B\left(\Gamma(\mathrm{Z_{p^2}})\right)$ is a complete graph namely K_{p-1}. The highest subset of $B\left(\Gamma(\mathrm{Z_{p^2}})\right)$ then $\{\mathrm{p, 2p, 3p, \dots. p(p-2)}\} \in \mathrm{S}$ implies $|S| = p - 2$ and the neighbourhood of the set S contains only one point $\{p(p-1)\}$ that is $|L(S)| = 1$. Clearly,

$$B\left(\Gamma(\mathrm{Z_{p^2}})\right) = \frac{|L(S)|}{|S|} = \frac{1}{p-1}$$

D. Theorem

Let p and q be the distinct prime numbers with $p < q$, then $B\left(\Gamma(\mathrm{Z_{pq}})\right) = \frac{p-1}{q-1}$.

Proof
The vertex set of $\Gamma(\mathrm{Z_{pq}})$ is

$$\{p, 2p, 3p, \dots, p(q-1), q, 2q, 3q, \dots, (p-1)q\}.$$

Let S and $L(S)$ be the lower degree set and the neighbourhood of S respectively.

Case (i)
Let $p = 2$, q is any prime > 2. We know that,

$$B\left(\Gamma(\mathrm{Z_{2q}})\right) = \frac{1}{q-1} = \frac{p-1}{q-1}.$$

Case(ii)
Let $p = 3$, q is any prime > 3.
The vertex set of $\Gamma(\mathrm{Z_{3q}})$ is $\{3,6,\dots,3(q-1),q,2q\}$. Let $u = q$ and $v = 2q$ be two vertices in $\Gamma(\mathrm{Z_{3q}})$ with higher degree then there exist any other vertex $\chi \neq q$ and $\chi \neq 2q$ in $\Gamma(\mathrm{Z_{3q}})$ such that χ is adjacent to both x and yy. That is, $x\chi = y\chi = 0$. But $xy \neq 0$. Therefore x and y are non-adjacent vertices.
Then the vertex set $V\left(\Gamma(\mathrm{Z_{3q}})\right)$ can be partitioned into two parts S and $L(S)$ such that $S = \{3,6,\dots.,3(q-1)\}$ and $N(S) = \{x, yy\} = \{q, 2q\}$.
Clearly $|S| = q - 1$ and $|L(S)| = 2$, then $\left|V\left(\Gamma(\mathrm{Z_{3q}})\right)\right| = |S| + |L(S)| = q - 1 + 2 = q + 1$..
Then, $b\left(\Gamma(\mathrm{Z_{3p}})\right) = \frac{|L(S)|}{|S|} = \frac{2}{q-1} = \frac{p-1}{q-1}, where\ p = 3\ and\ q > 3$.

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Case (iii) *Let* $p < q$.
The vertex set
$\Gamma(\mathrm{Z_{pq}}) = \{p, 2p, 3p, \dots, p(q-1), q, 2q, 3q, \dots, (p-1)q\}$.
Then,
$\left|V\left(\Gamma(\mathrm{Z_{pq}})\right)\right| = |S| + |L(S)| = p - 1 + q - 1 = p + q - 2$.
(i.e) $S = \{p, 2p, \dots, p(q-1)\}$ and
$L(S) = \{q, 2q, \dots, (p-1)q\}$.
Clearly, $d(x) < d(y)$ where $x \in S$ and $y \in L(S)$. Since, every vertex in S are adjacent to all the vertices in $L(S)$. Using all the above cases,

$$B\left(\Gamma(\mathrm{Z_{pq}})\right) = \frac{|L(S)|}{|S|} = \frac{p-1}{q-1}.$$

Similarly,
For any prime $p > 4$, $b\left(\Gamma(\mathrm{Z}_{4^n})\right) = \frac{15}{4^{n-1}-16}$.
Proof:
The vertex set of $\Gamma(\mathrm{Z}_{4^n})$ is $\{4,8,\dots,4(4^{n-1}-1)\}$ and $\left|V\left(\Gamma(\mathrm{Z}_{4^n})\right)\right| = 4^{n-1} - 1$ The proof is called the method of induction.
Case (i): Let $n = 5$.
The vertex set of $\Gamma(\mathrm{Z}_{256})$ is $\{4,8,\dots 252\}$ and $\left|V\left(\Gamma(\mathrm{Z}_{256})\right)\right| = 32$. Let S be the vertex subset of V and $N(S)$ be the neibourhood of S such that $d(u) < d(v)$ where $u \in S$ and $v \in N(S)$. Let $x = 48, y = 96$ and $x = 3$ then $ux = uy = 0$. This implies that the vertices 27 and 54 are adjacent to all the remaining vertices of $\Gamma(\mathrm{Z}_{256})$. Clearly,$27,54 \in N(S)$. Consider another vertex set $X = \{12,24,48,96,192,384\}$ which is the next maximum degree compared to the vertices 27,54. Let $x = 48$ and $y = 96$ then xy is divided by 256 that is x and y are adjacent. Hence,

$$b\left(\Gamma(\mathrm{Z}_{256})\right) = \frac{|L(S)|}{|S|} = \frac{15}{32} = \frac{15}{4^{5-1}-16} = \frac{15}{4^{n-1}-8}.$$

Case (ii): Let $n = 5$.
The vertex set of $\Gamma(\mathrm{Z}_{256})$ is $\{4,8,\dots 256\}$ and $\left|V\left(\Gamma(\mathrm{Z}_{243})\right)\right| = 80$. Using case(i), the vertex set $X = \{8,162\}$. Since, the vertices in X has highest degree then $X \in N(S)$. The vertex set $Y = \{27,54,108,135,189,216\}$ is the next maximum degree compared to the vertex set X. Let $u = 27$ and $v = 216$ in Y then uv is divided by 243 that is u and v are adjacent. Using case(i), any five vertices in Y belongs to $L(S)$. Thus, $L(S) = \{12,24,48,96,192,384\}$.
Case (iii) : Let $n > 5$.
In general, $\Gamma(\mathrm{Z}_{4^n})$ is $\{4,8,\dots,4(4^{n-1}-1)\}$

$$\left|V\left(\Gamma(\mathrm{Z}_{4^n})\right)\right| = 4^{n-1} - 1.$$

Clearly,

$$|S| = \left|V\left(\Gamma(\mathrm{Z}_{4^n})\right)\right| - |N(S)|$$
$$= 4^{n-1} - 1 - 15 = 4^{n-1} - 16$$

Hence,

$$B\left(\Gamma(\mathrm{Z}_{256})\right) = \frac{|L(S)|}{|S|} = \frac{15}{4^{n-1}-16}.$$

3. Conclusion

The zero-divisor graph seems to extract certain essential information relative to zero-divisors. The prime number of zero divisor graph B(Γ(Zn)) shows the maximum value of b(Γ(Zn)).

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