International Journal of Research in Engineering, Science and Management
Volume-2, Issue-11, November-2019
www.ijresm.com | ISSN (Online): 2581-5792

Derivation of Remainder Theorem for Polynomial of nth Degree

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***Abstract*: Remainder theorem has a significant contribution in algebraic mathematical calculations. This theorem deals with division of polynomial function of higher degree to that of linear term. Though, there has not been any solid proof available readily for deriving a general formula of polynomial function of nth degree. With the help of my mathematical formulation skills and higher engineering textbooks, I was able to derive proof for remainder theorem in case of nth degree polynomial being divided by linear factor (x-k). The approach started with deriving general equation for nth power polynomial and from there on using the general derived equation, quotient function is derived. Therefore, the entire derivation is split in two major parts; namely, derivation of general formula and derivation of quotient function. The approach of this process is by mathematical formulation.**

***Keywords*: linear factor, polynomial function, quotient function, remainder function.**

1. Introduction

- The polynomial function of nth degree in variable x can be expressed as,

$f(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots\ldots + a_2x^2 + a_1x + a_0$ (1)

Where, a_n, a_{n-1} , a_{n-2} , …. , a_2 are coefficients of variable x of respective powers.

- According to remainder theorem, when f(x) is divided by linear factor (x – k) the remainder left is f(k).

Expressing the remainder theorem mathematically we get,

$$\frac{f(x)}{x-k} = Q(X) + \frac{f(k)}{x-k}$$

Therefore, f(x) = (x-k) Q(x) + f(k) (2)

Where, Q(x) is quotient function and f(k) is remainder function

In this thesis report, we will derive first, the remainder theorem for nth degree polynomial and then we will derive the general form of quotient equation Q(x).

2. Derivation of general formula

$f(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots\ldots + a_2x^2 + a_1x + a_0$

$= a_nx^n - ka_nx^{n-1} + ka_nx^{n-1} + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots\ldots + a_2x^2 + a_1x + a_0$

$= a_nx^{n-1}(x-k) + (a_{n-1} + ka_n)x^{n-1} + a_{n-2}x^{n-2} + \ldots\ldots + a_2x^2 + a_1x + a_0$

$= a_nx^{n-1}(x-k) + (a_{n-1} + ka_n)x^{n-1} + a_{n-2}x^{n-2} + \ldots\ldots + a_2x^2 + a_1x + a_0$

$= a_nx^{n-1}(x-k) + (a_{n-1} + ka_n)x^{n-1} - k(a_{n-1} + ka_n)x^{n-2} + k(a_{n-1} + ka_n)x^{n-2} + a_{n-2}x^{n-2} + \ldots\ldots + a_2x^2 + a_1x + a_0$

$= a_nx^{n-1}(x-k) + (a_{n-1} + ka_n)x^{n-2}(x-k) + \{a_{n-2} + k(a_{n-1} + ka_n)\}x^{n-2} + \ldots\ldots + a_2x^2 + a_1x + a_0$

$= a_nx^{n-1}(x-k) + (a_{n-1} + ka_n)x^{n-2}(x-k) + (a_{n-2} + ka_{n-1} + k^2a_n)x^{n-2} + \ldots\ldots + a_2x^2 + a_1x + a_0$

In attempt to make terms which have (x-k) as factor to separate out quotient fanction, we can observe that coefficient of each term having factor (x-k) is now in steps of partial power series of k.

Similarly,

$f(x) = a_nx^{n-1}(x-k) + (a_{n-1} + ka_n)x^{n-2}(x-k) + (a_{n-2} + ka_{n-1} + k^2a_n)x^{n-2} + \ldots\ldots + a_2x^2 + a_1x + a_0$

$= a_nx^{n-1}(x-k) + \ldots\ldots + (a_1 + ka_2 + k^2a_3 + \ldots + k^{n-1}a_n)x + a_0$

$= a_nx^{n-1}(x-k) + \ldots + (a_1 + ka_2 + k^2a_3 + \ldots + k^{n-1}a_n)x - k(a_1 + ka_2 + k^2a_3 + \ldots + k^{n-1}a_n) + k(a_1 + ka_2 + k^2a_3 + \ldots + k^{n-1}a_n) + a_0$

$= a_nx^{n-1}(x-k) + \ldots + (a_1 + ka_2 + k^2a_3 + \ldots + k^{n-1}a_n)(x-k) + [a_0 + a_1k + a_2k^2 + a_3k^3 + \ldots\ldots + a_{n-1}k^{n-1} + a_nk^n]$

$= a_nx^{n-1}(x-k) + (a_{n-1} + ka_n)x^{n-2}(x-k) + (a_{n-2} + ka_{n-1} + k^2a_n)x^{n-3}(x-k) + \ldots + (a_1 + ka_2 + k^2a_3 + \ldots + k^{n-1}a_n)(x-k) + f(k)$

Because, $[a_0 + a_1k + a_2k^2 + a_3k^3 + \ldots\ldots + a_{n-1}k^{n-1} + a_nk^n] = f(k)$

Hence,

$f(x) = [a_nx^{n-1} + (a_{n-1} + ka_n)x^{n-2} + (a_{n-2} + ka_{n-1} + k^2a_n)x^{n-3} + \ldots + (a_1 + ka_2 + k^2a_3 + \ldots + k^{n-1}a_n)](x-k) + f(k)$ (3)

Comparing this result with standard form f(x) = (x-k) Q(x) + f(k), we can observe that,

$Q(x) = a_nx^{n-1} + (a_{n-1} + ka_n)x^{n-2} + (a_{n-2} + ka_{n-1} + k^2a_n)x^{n-3} + \ldots + (a_1 + ka_2 + k^2a_3 + \ldots + k^{n-1}a_n)$ (4)

3. Derivation of quotient function

As we have derived the equation of remainder theorem, we can use that equation to obtain formula for quotient function Q(x).

as,

$f(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots\ldots + a_2x^2 + a_1x + a_0$

$= a_nx^n - a_nk^n + a_{n-1}x^{n-1} - a_{n-1}k^{n-1} + a_{n-2}x^{n-2} - a_{n-2}k^{n-2} + \ldots + a_2x^2 - a_2k^2 + a_1x - a_1x + a_0 + + a_1k + a_2k^2 + a_3k^3 + \ldots\ldots + a_{n-1}k^{n-1} + a_nk^n$

(adding and subtracting terms $a_n k^n$, $a_{n-1}k^{n-1}$, $a_{n-2}k^{n-2}$,…….., a_2k^2 ,a_1k)

Hence,

$f(x)=a_n(x^n-k^n)+a_{n-1}(x^{n-1}-k^{n-1})+a_{n-2}(x^{n-2}-k^{n-2})+..+a_2(x^2-k^2)+a_1(x-k)+[a_0+a_1k+a_2k^2+a_3k^3+..+a_{n-1}k^{n-1}+a_nk^n]$

$=a_n(x^n-k^n)+a_{n-1}(x^{n-1}- k^{n-1})+a_{n-2}(x^{n-2}-k^{n-2})+..+a_2(x^2-k^2)+a_1(x-k) + f(k)$ (5)

Therefore, $(x-k).Q(x) = a_n(x^n-k^n)+a_{n-1}(x^{n-1}- k^{n-1})+a_{n-2}(x^{n-2}-k^{n-2})+..+a_2(x^2-k^2)+a_1(x-k)$ (6)

Here,

$x^n-k^n = x^n - kx^{n-1} + kx^{n-1} – k^2x^{n-2} + k^2x^{n-2}-…...- k^{n-2}x^2+ k^{n-2}x^2 – k^{n-1}x + k^{n-1}x - k^n$

$= x^{n-1}(x-k) + kx^{n-2} (x-k) + k^2x^{n-3}(x-k) +…..+k^{n-2}x(x-k) + k^{n-1}(x-k)$

$= (x^{n-1} + kx^{n-2} + k^2x^{n-3} +……+ k^{n-2}x + k^{n-1})(x-k)$

Similarly,

$x^{n-1}- k^{n-1} = (x^{n=2} + kx^{n-3} + k^2x^{n-4} +….+ k^{n-2})(x-k)$

$x^{n-2} – k^{n-2} = (x^{n-3} + kx^{n-4} +…..+ k^{n-3})(x-k)$

$x^{n-3}- k^{n-3} = (x^{n-4} + kx^{n-5}+…..+k^{n-4})(x-k)$

$x^2- k^2 = (x+k)(x-k)$

putting all these values in equation (6) we get,

$(x-k)Q(x) = a_n(x^{n-1} + kx^{n-2} + k^2x^{n-3} +……+ k^{n-2}x + k^{n-1})(x-k) + a_{n-1}(x^{n=2} + kx^{n-3} + k^2x^{n-4} +….+ k^{n-2})(x-k) + a_{n-2}(x^{n-3} + kx^{n-4} +…..+ k^{n-3})(x-k) +….+ a_2(x+K)(x-k) + a_1(x-k)$

Therefore,

$Q(x)= [a_n(x^{n-1} + kx^{n-2} + k^2x^{n-3} +……+ k^{n-2}x + k^{n-1})+a_{n-1}(x^{n-2} + kx^{n-3} + k^2x^{n-4} +….+ k^{n-2})+…+ a_2(x+K)+ a_1]$ (7)

Which is the desired form of quotient function.

4. Conclusion

Hence, this paper presented derivation of remainder theorem for polynomial of n^{th} degree.

References

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