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# Comparative Study of Runge-Kutta Method of Order 4 And 6

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***Abstract*: In this paper we broach about Runge-Kutta of order 4 and 6 with differential equations and its application. The differential equation problems are solved to find their approximation accuracy and error difference. The result and its comparison are tabulated. From the probe we heed that the Runge-Kutta method of order of 4 gives the preferred accuracy and coded with Matlab.**

***Keywords*: Runge-Kutta method, Initial Value Problems (IVP), Error Analysis, Euler's method, Problem Formulation, Matlab software.**

## 1. Introduction

Differential Equations are of great help for solving complex mathematical problems in almost every section of Engineering, Science and Mathematics. In mathematics a number of real problems arise in the form of differential equations. These differential equations are either in the form of ordinary differential equation or partial differential equation. Usually, most of the problems which are modelled by these differential equations are so complicated that it is hard to determine the exact solution and one of two approaches is taken to approximate the solution. The first technique we will use is reducing the differential equations into a form which can be solved exactly and the results can be used to approximate the solution of the original problems. Another technique that we will use in this article is the approximation method which gives a more perfect result and less relative error. To solve those mathematical problems where it is very difficult or nearly impossible to determine the exact solution, numerical methods are used. Only a limited number of differential equations can be solved analytically. The solutions of a large number of differential equations cannot be determined using the familiar analytical methods. So in these cases we need to apply numerical methods to solve a differential equation under certain initial restriction or restrictions. To find the solution of initial value problem of ordinary differential equations there are a number of numerical methods. In this research paper we will present Runge-Kutta method to find the solution of initial value problems for ordinary differential equations.

## 2. Runge Kutta method

In numerical analysis, the Runge–Kutta methods are a family of implicit and explicit iterative methods, which include the well-known routine called the Euler Method, used in temporal discretization for the approximate solutions of ordinary differential equations. These methods were developed around 1900 by the German mathematicians Carl Runge and Wilhelm Kutta, Runge.

## 3. Uses of Runge-Kutta method

The Runge-Kutta Method is a numerical integration technique which provides a better approximation to the equation of motion. Unlike the Euler's Method, which calculates one slope at an interval, the Runge-Kutta calculates four different slopes and uses them as weighted averages. Kutta method was designed to give greater accuracy with advantage of requiring only the functional values of some selected points in sub – interval.

## 4. Derivation and method

The Runge kutta method is based on the following equation,

$$y_{i+1} = ( y_i + a_1k_1 + a_2k_2 + a_3k_3 + a_4k_3 ) \qquad (1)$$

Where knowing the value of $y = y_i$ at $x_i$ , we can find the value of $y = y_{i+1}$ at $x_{i+1}$ and $h = x_{i+1} - x_i$

Equation is equated to the first five terms of Taylor series

$$y_{i+1}=y_i + \frac{h}{1!}y'_0 + \frac{h^2}{2!}y''_0 + \frac{h^3}{3!}y'''_0 + \frac{h^4}{4!}y'^{v}_0 + \frac{h^5}{5!}y^{v}_0 \qquad (2)$$

Knowing that $\frac{dy}{dx}$ = f(x,y) and $x_{i+1} - x_i = h$

$$y_{i+1} = y_i + f(x_i, y_i)h + \frac{1}{2!} f'(x_i, y_i)h^2 + \frac{1}{3!} f'(x_i, y_i)h^3 + \frac{1}{4!} f''(x_i, y_i)h^4 \qquad (3)$$

Based on equating equation (2) and equation (3), one of the popular solutions used is to get $4^{th}$ and $6^{th}$

*Fourth Order:*

$Y_{i+1} = y_i + 1/6\,(k_1 + 2k_2 + 2k_3 + k_4)\,h$

$k_1 = f(x_i, y_i)$

$k_2 = f(x_i + \frac{h}{2}, y_i + \frac{k_1}{2} h)$

$k_3 = f(x_i + \frac{h}{2}, y_i + \frac{k_2}{2})$

$k_4 = f(x_i + h, y_i + k_3)$

*Application of fourth order:*

- Fuzzy differential equation has become an interesting field these days and the fourth order R-K method has been found as a usual technique to unravel such as fuzzy differential equation (parandin, 2014).
- An algorithm to solve fuzzy initial value problem based on fourth order R-K method is proposed in detail by (Sharmila and Amritharaj, 2013). It was established that in the new method, order of convergence is $O(h^2)$. This was observed that the proposed method suited very well to solve linear and non –linear initial value problems.

*Remarks:*

- To achieve the desired accuracy the step size has to be very small.
- The numerical results obtained agreed with exact solutions to good extent.

*Sixth Order:*

$Y_{i+1} = y_i + \frac{1}{90}(7k_1 + 2k_2 + 2k_3 + k_4)\,h$

$k_1 = hf(x_i, y_i)$

$k_2 = hf(x_i + \frac{h}{2}, y_i + \frac{k_1}{2})$

$k_3 = hf[x_i + \frac{h}{4}, y_i + \frac{1}{16}(3k_1 + k_2)$

$k_4 = hf(x_i + \frac{h}{2}, y_i + \frac{k_3}{2})$

$k_5 = hf([x_i + \frac{3h}{4}, y_i + \frac{1}{16} + (-3k_2 + 6k_3 + 8k_4)]$

$k_6 = hf[x_i + h, y_i + \frac{1}{7} + (k_1 + 4k_2 + 6k_3 - 12k_4 + 8k_5)]$

*Application of Sixth order of R-K method:*

An efficient sixth order three implicit R-K method has been by Agama and Yahaya to solve initial value problem of first order (Agama and Yahaya, 2014). To derive continuous schemes collocation method was used which has given higher order schemes. In this algorithm types of implicit approaches under discussion were,

- Singly implicit approach with order $(p = s)$ ( s = number of stages)
- Diagonally implicit methods
- Multiply implicit methods with order $(p = 2s)$

*Remarks:*

- Implicit R-K approaches based on Gaussian quadrature have order $= 2s$ , s = number of stages.
- The proper scheme satisfies Runge-Kutta conditions $\sum_{j=0}^{3} a_{ij} = c_i$ , $\sum_{i=1}^{3} b_i = 1$.

## 5. Problem formulation

In this segment we consider three numerical methods for obtaining the numerical solutions of the initial value problem (IVP) of the first-order ordinary differential equation is of the form

$$y' = f(x, y(x)),\ x\epsilon(x0, xn),\ y(x0)$$

where $' = dy/dx$ and $f$(x, y(x)) is the given function and y(x) is the solution of the equation (1). In this article, we find the solution of this equation on a finite interval $(x0, xn)$, starting with the initial point $x0$. A continuous approximation to the solution y(x) will not be obtained; instead, approximations to y will be generated at various values, called mesh points, in the interval $(x0,)$. Numerical methods employ the Equation (1) to obtain approximations to the values of the solution corresponding to various selected values of the solution corresponding to different selected values of, n = 0,1,2, … … . The parameter h is called the step size. The numerical solution of (1) is given by a set of points $\{(x0, xn)$: n = 0,1,2,3, … … n} and each point $(x0, xn)$ is an approximation to the corresponding point $\{xn, y(xn)\}$ on the solution curve.

## 6. Error analysis

There exist two kinds of errors in numerical solution of ordinary differential equations. Round-off errors and Truncation errors happen when ordinary differential equations are solved numerically. Rounding errors initiate from the verity that computers can only characterize numbers using a fixed and restricted number of important figures. Thus, such numbers cannot be represented accurately in computer memory. The inconsistency introduced by this restriction is call Round-off error. Truncation errors in numerical study occur when approximations are used to determine a number of quantities. The exactness of the solution will rely on how miniature we make the step size, h. A numerical method is said to be convergent if ,

$$\lim h\rightarrow 0\ max1\leq n\leq N\ y\ xn - yn$$

Where y (xn) denotes the approximate solution and yn denotes the exact solution. In this work we consider two initial value problems to examine accuracy of the proposed methods. The Approximated solution is determined by using MATLAB software for three proposed numerical methods at different step size. The maximum error is defined by,

$$ER = \lim h\rightarrow 0\ \ x1\leq n\leq N\ y\ xn - yn$$

## 7. Numerical examples

To apply the$4^{th}$,$6^{th}$ order of Runge Kutta method to find y (0.2) given that $y' = x + y$, $y(0) = 1$.

*Proof:*

Result for the numerical solutions of $y' = x + y$ and $y(0) = 1$

$x_0 = 0, y_0 = 1$
By $4^{th}$ Order:
$k_1 = \text{f}(\ x_i, y_i)$
$k_1 = 0.1$
$k_2 = \text{f}(\ x_i + \frac{h}{2}, y_i + \frac{k_1}{2}\text{h})$
$k_2 = 0.11$
$k_3 = \text{f}(\ x_i + \frac{h}{2}, y_i + \frac{k_2}{2})$
$k_3 = 0.1105$
$k_4 = \text{f}(\ x_i + \text{h}, y_i + k_3)$
$k_4 = 0.12105$
$Y_{i+1} = y_i + 1/6\,(k_1 + 2k_2 + 2k_3 + k_4)\,\text{h}$
$y(0.2) = 1.2428$
Similarly, the values of
0.4 = 1.5836
0.6 = 2.0442
0.8 = 2.6510
1 = 3.4366
By $6^{th}$ Order:
$k_1 = \text{hf}(\ x_i, y_i)$
$k_1 = 0.2$
$k_2 = \text{hf}(\ x_i + \frac{h}{2}, y_i + \frac{k_1}{2})$
$k_2 = 0.24$
$k_3 = \text{hf}[\ x_i + \frac{h}{4}, y_i + \frac{1}{16}(3k_1 + k_2)$
$k_3 = 2.13$
$k_4 = \text{hf}(\ x_i + \frac{h}{2}, y_i + \frac{k_3}{2})$
$k_4 = 0.433$
$k_5 = \text{hf}([\ x_i + \frac{3h}{4}, y_i + \frac{1}{16} + (-3k_2 + 6k_3 + 8k_4)\ ]$
$k_5 = 0.4241$
$k_6 = \text{hf}[\ x_i + h, y_i + \frac{1}{7} + (k_1 + 4k_2 + 6k_3 - 12k_4 + 8k_5)\ ]$
$k_6 = 0.4898$
$Y_{i+1} = y_i + \frac{1}{90}(7k_1 + 2k_2 + 2k_3 + k_4)\,\text{h}$
$y(0.2) = 2.0195$
Similarly, the values of
0.4 = 3.3241
0.6 = 6.0394
0.8 = 7.2780
1 = 2.3324

## 8. First order ordinary differential equation solution for Runge-Kutta method of exact solution and error value

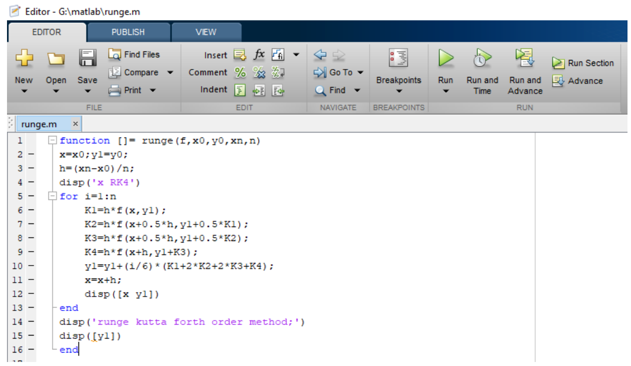
Table 1
The Error analysis and Approximation solution for step size h=0.2

| X | Runge Kutta Order 4 | Runge Kutta Order 6 | Exact Solution | Error Value (4) | Error Value (6) |
|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 0 | 0 |
| 0.2 | 1.2428 | 2.0195 | 1.2428 | 0 | -0.7767 |
| 0.4 | 1.5836 | 3.3241 | 1.5836 | 0 | 1.7405 |
| 0.6 | 2.0442 | 6.0394 | 2.0442 | 0 | 3.9952 |
| 0.8 | 2.6510 | 7.6420 | 2.6510 | 0 | 4.991 |
| 1 | 3.4365 | 8.3324 | 3.4366 | 0.006 | 4.8958 |

## 9. Runge-Kutta method of order 4 and 6 solution using Matlab coding

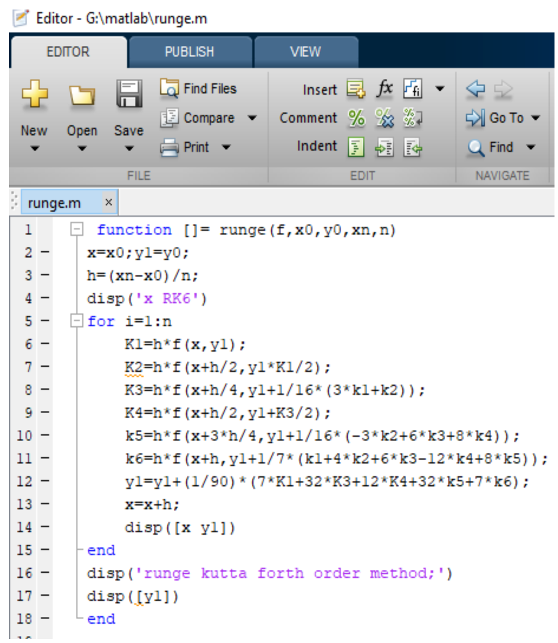
*Problem:*
Solve $y' = x + y$, given $y(0) = 1$, and get $y(1)$ by RUNGE - KUTTA method.

*By order 4:*




```
function []= runge(f,x0,y0,xn,n)
x=x0;y1=y0;
h=(xn-x0)/n;
disp('x RK4')
for i=1:n
    K1=h*f(x,y1);
    K2=h*f(x+0.5*h,y1+0.5*K1);
    K3=h*f(x+0.5*h,y1+0.5*K2);
    K4=h*f(x+h,y1+K3);
    y1=y1+(i/6)*(K1+2*K2+2*K3+K4);
    x=x+h;
    disp([x y1])
end
disp('runge kutta forth order method;')
disp([y1])
end
```

*By order 6:*




```
function []= runge(f,x0,y0,xn,n)
x=x0;y1=y0;
h=(xn-x0)/n;
disp('x RK6')
for i=1:n
    K1=h*f(x,y1);
    K2=h*f(x+h/2,y1*K1/2);
    K3=h*f(x+h/4,y1+1/16*(3*k1+k2));
    K4=h*f(x+h/2,y1+K3/2);
    k5=h*f(x+3*h/4,y1+1/16*(-3*k2+6*k3+8*k4));
    k6=h*f(x+h,y1+1/7*(k1+4*k2+6*k3-12*k4+8*k5));
    y1=y1+(1/90)*(7*K1+32*K3+12*K4+32*k5+7*k6);
    x=x+h;
    disp([x y1])
end
disp('runge kutta forth order method;')
disp([y1])
end
```

## 10. Conclusion

In this paper, we obtain the approximation solution & Exact solution for the ordinary differential equation with initial value condition using Mat-lab for Runge-Kutta method of order 4 and 6.

Finding more accurate results needs the step size smaller for all methods. Comparing the results of two order. From the study, the Runge-Kutta method of order 4 was found to be generally more accurate and also the approximate solution converged faster to the exact solution as shown. The accuracy of solution of order 4 is nearer and error value is also very less.

The result obtained through programming for Runge-Kutta method is exact, effective, and making it easy to bypass analytical complex calculation.

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