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A Study on Borel Measures for Hardy Spaces

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***Abstract*—In this paper, we investgate the theory of Hardy spaces and its related theorems are discussed. The invariant subspaces of the borel measures for hardy spaces are also proved.**

***Index Terms*—Hardy Spaces, invariant subspace, Borel measure**

I. Introduction

Let **T** denotes the unit circle in the complex plane and μ is a lebesgue measure on **T** normalized so that $\mu(\boldsymbol{T}) = 1$

Let $L^P(\boldsymbol{T})$ be sequence of measurable functions on the space **T**. $L^P(\boldsymbol{T})$ is also a lebesgue space with respect to measure μ.

The Hardy will be defined as a closed subspace is $L^P(\boldsymbol{T})$. In the case of p=1 or ∞

For n in **Z** let χ_n denote the function on T defined $\chi_n(z) = z^n$.
If we define

$$H' = \{f \in L'(T): \frac{1}{2\pi}\int_0^{2\pi} f\chi_n dt = 0 \quad for\ n = 1,2,3, \dots.\}$$

then H^1 is obviously a linear subspace of $L'(\boldsymbol{T})$
Moreover, since the set

$$\{f \in L'(T): \frac{1}{2\pi}\int_0^{2\pi} f\chi_n dt = 0\}$$

is the kernel of bounded linear functional on $L'(\boldsymbol{T})$, hence H' is a closed subspace of $L'(\boldsymbol{T})$ and hence a Banach space.
For $P = \infty$

$$H^\infty = \{\phi \in L^\infty(T): \frac{1}{2\pi}\int_0^{2\pi} \phi\chi_n dt = 0 \quad for\ n = 1,2,3, \dots.\}$$

is closed subspace of $L^\infty(T)$.

Moreover in the case

$$\{\phi \in L^\infty(T): \frac{1}{2\pi}\int_0^{2\pi} \phi\chi_n dt = 0\}$$

In general for P=1,2,..∞

$$H^p = \{f \in L^p(T): \int_0^{2\pi} f(e^{i\varphi})\chi_n(e^{i\varphi})d\varphi = 0 \quad for\ n > 0\}$$

Hence H^p is closed subspace of $L^p(T)$ and H^p is Banach space.

II. Mathematical Formulation

Proposition : 2.1
If φ is in $L^\infty(T)$, then H^2 is an invariant subspace for φ, if and only if φ is in H^∞.

Proof:

Let M_ϕ is the multiplication operator defined by $M_\phi f = \phi f$ for f in $L^2(T)$.
If $M_\phi H^2$ is contained in H^2 since 1 is in H^2 and hence ϕ is in H^∞.
Conversely,
If ϕ is in H^∞
Then $\phi\mathcal{P}_+$is contained in H^2
Since for $P = \sum_{j=0}^N \propto_j \chi_j$ in $\mathcal{P}_+$
We have

$$\int_0^{2\pi} (\phi P)\chi_n d\varphi = \sum_{j=0}^N \propto_j \int_0^{2\pi} \phi\chi_{j+n} d\varphi = 0 \quad for\ n > 0$$

Since H^2 is the closure of $\mathcal{P}_+$, we have ϕH^2 contained in H^2 which completes the proof.

Corollary: 2.2
The space H^∞ is an algebra.
Proof:
If ϕ and ψ are in H^∞
Then $M_{\phi\psi}H^2 = M_\varphi(M_\psi H^2) \subset M_\varphi H^2 \subset H^2$
By the above proposition
H^2 is invariant subspace of H^∞then implies that $\varphi\psi$ is in H^∞
Hence $H^\infty$is an algebra.

Theorem:2.3
If μ is in the space M(**T**) of Borel measures on **T** and $d\mu = 0$ for n in **Z** then $\mu = 0$.

Proof :
Since the linear span of the functions $\{\chi_n\}_{n\in z}$ is uniformly dense C(**T**) and M(**T**) is the dual of C(**T**), the measure μ represents the zero functional and hence must be the zero measure.

Corollary:2.4
If f is a function in L(T)
Such that $\int_0^{2\pi} f(e^{i\varphi})\chi_n(e^{i\varphi})d\varphi = 0 \quad for\ n\ in\ z.$
$f = 0$ a.e.

Proof :
If we define the measure μ on **T** such that

$$\mu(E) = \int_E^{\infty} f(e^{i\varphi})d\varphi$$

By hypothesis become $\int_T^0 \chi_n d\mu = 0 \quad for\ n\ in\ z$

Hence $\mu = 0$ and $f = 0$ a.e

Corollary: 2.5

If f is a real-valued function in H^1, then $f = \propto$ a.e for some $\propto$ in R.

Proof :

If we set $\propto = (\frac{1}{2\pi}) \int_0^{2\pi} f(e^{i\varphi})d\varphi$

Then $\propto$ is real and

$$\int_0^{2\pi} (f - \propto)\chi_n d\varphi = 0 \quad for \quad n \geq 0$$

Since $f - \propto$ is real valued; taking the complex conjugate of the proceeding equation yields.

$$\int_0^{2\pi} (f - \propto)\overline{\chi_n}\, d\varphi = \int_0^{2\pi} (f - \propto)\chi_{-n} d\varphi = 0 \qquad for\ n \geq 0$$

Combining this with previous identity yields.

$$\int_0^{2\pi} (f - \propto)\chi_n d\varphi = 0 \quad for\ all\ n.$$

Hence $f = \propto$ a.e.

Corollary: 2.6

If both f and $\bar{f}$ are in H^1, then $f = \propto$ a.e. for some $\propto$ in C.

Proof:

Apply previous corollary to the real valued functions

$$\tfrac{1}{2}(f + \bar{f}) \ \ and\ \tfrac{1}{2}(f + \bar{f})/i$$

Given:

f and its conjugate $\bar{f}$ are in H^1

$\Rightarrow \frac{1}{2}(f + \bar{f})$ and $\frac{1}{2}(f + \bar{f})/i$ belong to H^1

Let $\propto = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\varphi})d\varphi$ then $\propto$ is complex

And $\int_0^{2\pi} \left(\frac{1}{2}(f + \bar{f}) - \propto\right)\chi_n d\varphi = 0 \qquad for\ n \geq 0$

taking complex conjugate

$$\int_0^{2\pi} \left(\frac{1}{2}(f + \bar{f})/i - \propto\right)\overline{\chi_n} d\varphi = 0 \qquad for\ n \geq 0$$

combining above two results

$$\frac{1}{2}(f + \bar{f}) = \propto \qquad f + \bar{f} = 2\propto \Rightarrow 2f = 2\propto (if\ f = \bar{f})$$

$$f = \propto \text{ a.e.} \quad \text{where } \propto \in C \qquad f \in H'$$

Borel Theorem: 2.7

If $\mathcal{M}$ is a positive Borel measure on **T**, then a closed subspace $\mathcal{M}$ of $L^2(\mu)$ satisfies $\chi_1\mu = \mathcal{M}$ if and only if there exist a Borel subset E of **T** such that

$$= L_E^2(\mu) = \{f \in L^2(\mu): f(e^{it}) = 0 \qquad for\ \ e^{it} \in E\}$$

Proof:

If $M = L_E^2(\mu)$ then clearly $\chi_1\mu = \mathcal{M}$

Conversely, if $\chi_1\mu = \mathcal{M}$ then it follows that $\mathcal{M} = \chi_{-1}\chi_1 = \chi_{-1}$ and hence $\mathcal{M}$ is a reducing subspace for the operator $\mathcal{M}_{\chi 1}$ on $L^2(\mu)$.Therefore if F denotes the projection on to $\mathcal{M}$, then F commutes with $\mathcal{M}_{X1}$

By proposition:

"If T is an operator on $\mathcal{H}$, $\mathcal{M}$ is closed subspace of $\mathcal{H}$ and $P_{\mathcal{M}}$ is a projection on to $\mathcal{M}$ then $\mathcal{M}$ is and invariant subspace for T, if and only if $P_{\mathcal{M}}TP_{\mathcal{M}} = TP_{\mathcal{M}}$ if and only if $\mathcal{M}^1$ is an invariant subspace for T^* further, $\mathcal{M}$ is a reducing subspace for T if and only if $P_{\mathcal{M}}T = TP_{\mathcal{M}}$ and only if $\mathcal{M}$ is an variant subspace for both T and T^* with M_ϕ for ϕ in C(T)

By corollary if x is a compact Hausdroff space and μ is a finite positive regular Borel measure on X, then C(X) is W^*- dense in $L^\infty(\mu)$

The algebra $m = \{M_\phi \phi \in L^\infty(\mu)\}$ is maximal abelian

Now conclude that F is of the form M_ϕ for some ϕ in $L^\infty(\mu)$

Hence $M = \{f \in L^2(\mu): f(e^{it}) = 0$ for $e^{it} \in$ E}

Theorem: 2.8

If μ is a positive Borel measure on **T**, then a nontrivial closed subspace $\mathcal{M}$ of $L^2(\mu)$ satisfies $\chi_1\mathcal{M} \subset \mathcal{M}$ and $\bigcap_{n\geq 0}\chi_n\mathcal{M} = \{0\}$ if and only if there exist a Borel function ϕ such that $|\phi|^2 d\mu = {d\phi}/{2\pi}$ then the function $\psi f = \varphi f$ is μ measurable for f in H^2 and

$$||\psi f||_2^2 = \int_T^0 ||\varphi f||^2 d\mu = \frac{1}{2\pi}\int_0^{2\pi} |f|^2 d\varphi = ||f||_2^2$$

φ has the image $\mathcal{M}$ of H^2 under the isometric ψ is a closed subspace $L^2(\mu)$. μ is invariant for M_χ.

Since $\chi_1(\psi f) = \psi(\chi_1 f)$ then we have

$\bigcap_{n\geq 0}\chi_n\mathcal{M} = \psi[\bigcap_{n\geq 0}\chi_n H^2] = \{0\}$

Hence $\mathcal{M}$ is a simply invariant subspace for $M_{\chi 1}$ conversely suppose $\mathcal{M}$ is a nontrivial closed invariant subspace for which M_χ.

$\bigcap_{n\geq 0}\chi_n\mathcal{M} = \{0\}$.

Then L=$\mathcal{M} \bigcirc$ $\chi_n\mathcal{M}$ is non trivial and $\mathcal{M} = \chi_n\mathcal{M} \bigcirc \chi_{n+1}\mathcal{M}$ since multiplication by χ_1 is an isometry on $L^2(\mathcal{M})$ therefore, the subspace $\sum_{n=0}^{\infty} \oplus \chi_n \mathrm{L}$ is contained in μ and an easy argument reveals

$\mathcal{M}$ - $(\sum_{n=0}^{\infty} \oplus \chi_n \mathrm{L})$ to be $\bigcap_{n\geq 0}\chi_n\mathcal{M}$ and hence $\{0\}$

If φ is a unit vector in L, then φ is n orthogonal to $\chi_n\mathcal{M}$ and hence to $\chi_n\varphi,\ n > 0$ and thus we have $\varphi = (\varphi, \chi_n\varphi) = \int_T^0 |\varphi|^2 \chi_n d\mu$ for n>0

We see that $|\varphi|^2 d\mu = {d\varphi}/{2\pi}$

Suppose L has dimension greater than one and φ^1 is a unit vector in L orthogonal to φ. In this case we have.

$0 = (\chi_n\varphi, \chi_m\varphi') = \int_T^0 \varphi\overline{\varphi^1}\chi_{n-m} d\mu \qquad for\ n,m \geq 0$

Thus $\int_T \chi_k dv = 0 \qquad for\ k\ in\ z$

Where $dv = \varphi\overline{\varphi'}d\mu$. Therefore $\varphi\overline{\varphi'}d\mu = 0\mu$ a.e combining this with that $|\varphi|^2 d\mu = |\varphi^1|^2 d\mu$ leads to a contraction and hence L is one dimensional. Thus we obtain that $\varphi\mathcal{P}_+$is dense and hence $\mathcal{M}= \varphi H^2$hence the proof.

Beurling Theorem:2.9

A function φ in H^∞ is an inner function if $|\varphi| = 1$ a.e. If $T_{\chi 1} = M_{\chi 1}|H^2$, then a nontrivial closed subspace $\mathcal{M}$ of H^2 is invariant for $T_{\chi 1}$ if and only if $\mathcal{M}= \varphi H^2$for some inner function P.

Proof:

If φ is an inner function, then $\varphi\mathcal{P}_+$is contained in H^∞, since the later is an algebra and is therefore contained in H^2, since φH^2 is the closure of $\varphi\mathcal{P}_+$

We see that φH^2 is a closed invariant subspace for $T_{\chi 1}$

Conversely, if $\mathcal{M}$ is a nontrival closed invariant subspace for $T_{\chi 1}$, then $\mathcal{M}$ satisfies the hypotheses of the proceeding theorem for

$d\mu = {d\varphi}/{2\pi}$ and hence there exists a measurable function φ such that $\mu = \varphi H^2$and $|\varphi|^2\, {d\varphi}/{2\pi} = {d\varphi}/{2\pi}$

Therefore $|\varphi| = 1\ a.e.$ since 1 is in H^2

We get $\varphi = \varphi.1$ is in H^2

Thus φ is an inner function.

Theorem: 2.10

If μ is a positive Borel measure on **T** then a closed invariant subspace $\mathcal{M}$ for $M_{\chi 1}$ has a unique direct sum decomposition $\mathcal{M}=\mathcal{M}_1\oplus\mathcal{M}_2$ such that each $\mathcal{M}_1$ and $\mathcal{M}_2$is invariant for $M_{\chi 1}$, $\chi_1\mathcal{M}_1=\mathcal{M}_1$and $\bigcap_{n\geq 0}\chi_n\mathcal{M}=\{0\}$

Proof:

If we set $\mathcal{M}_1= \bigcup_{n\geq 0}\chi_n\mathcal{M}$ then $\mathcal{M}_1$ is a closed invariant subspace for $M_{\chi 1}$ satisfying $\chi_1\mathcal{M}_1=\mathcal{M}_1$the function f in $\mathcal{M}$ if and only if it can be written in the form $\chi_n g$ for some g in $\mathcal{M}$ for each n>0.

If we set $\mathcal{M}_2=\mathcal{M}-\mathcal{M}_1$

Then function f in $\mathcal{M}$ is in $\mathcal{M}_2$, if and only if (f,g)=0 for all g in $\mathcal{M}_1$

Since $0 = (f, g) = (\chi_1 f, \chi_1 g)$ and $\chi_1\mathcal{M}_1=\mathcal{M}_1$it follows that $\chi_1 f$ is in $\mathcal{M}_2$ and hence $\mathcal{M}_2$ is invariant for $M_{\chi 1}$ if f is in $\bigcap_{n\geq 0}\chi_n\mathcal{M}_2$, then it is in $\mathcal{M}_1$ and hence f=0 hence the proof.

Theorem: 1.5

If f is a nonzero function in H^2, then the set $\{e^{it} \in T : f(e^{it}) = 0\}$ has measure zero.

Proof :

Set $E = \{e^{it} \in T : f(e^{it}) = 0\}$ and define

$\mathcal{M} = \{g \in H^2 : g(e^{it}) = 0 \quad for\ e^{it} \in E\}$

it is clear that $\mathcal{M}$ is a closed invariant subspace for $T_{\chi 1}$ which is nontrival since f is in it.

By beurlings theorem there exist inner function φ where $|\varphi| = 1\ a.e$. Such that $\mathcal{M}= \varphi H^2$since 1 is in H^2 it follows that φ is in $\mathcal{M}$ and hence that E is contained in $\{e^{it} \in T: \varphi(e^{it}) = 0\}$. Since $|\varphi| = 1\ a.e.$ Hence the measure zero.

III. Conclusion

The theory of Hardy spaces and its related theorems are discussed. The invariant subspace of the borel measures for hardy spaces are also proved.

References

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