Abstract—In this paper, we investigate the theory of Hardy spaces and its related theorems are discussed. The invariant subspaces of the borel measures for hardy spaces are also proved.

Index Terms—Hardy Spaces, invariant subspace, Borel measure

I. INTRODUCTION

Let \( T \) denotes the unit circle in the complex plane and \( \mu \) is a lebesgue measure on \( T \) normalized so that \( \mu(T) = 1 \). Let \( L^p(T) \) be sequence of measurable functions on the space \( T \). \( L^p(T) \) is also a lebesgue space with respect to measure \( \mu \).

The Hardy will be defined as a closed subspace is \( L^p(T) \). In the case of \( p=1 \) or \( \infty \)

For \( n \in \mathbb{Z} \) let \( \chi_n \) denote the function on \( T \) defined \( \chi_n(z) = z^n \).

If we define

\[
H' = \{ f \in L'(T) : \frac{1}{2\pi} \int_0^{2\pi} f(z) \chi_n(z) \, dt = 0 \quad \text{for} \quad n = 1, 2, 3, \ldots \}
\]

then \( H^1 \) is obviously a linear subspace of \( L'(T) \).

Moreover, since the set

\[
\{ f \in L'(T) : \frac{1}{2\pi} \int_0^{2\pi} f(z) \chi_n(z) \, dt = 0 \}
\]

is the kernel of bounded linear functional on \( L'(T) \), hence \( H' \) is a closed subspace of \( L'(T) \) and hence a Banach space.

For \( P = \infty \)

\[
H^\infty = \{ \phi \in L^\infty(T) : \frac{1}{2\pi} \int_0^{2\pi} \phi(z) \chi_n(z) \, dt = 0 \quad \text{for} \quad n = 1, 2, 3, \ldots \}
\]

is closed subspace of \( L^\infty(T) \).

Moreover in the case

\[
\{ \phi \in L^\infty(T) : \frac{1}{2\pi} \int_0^{2\pi} \phi(z) \chi_n(z) \, dt = 0 \}
\]

In general for \( P=1,2,\ldots,\infty \)

\[
H^p = \{ f \in L^p(T) : \int_0^{2\pi} f(e^{i\theta}) \chi_n(e^{i\theta}) \, d\theta = 0 \quad \text{for} \quad n > 0 \}
\]

Hence \( H^p \) is closed subspace of \( L^p(T) \) and \( H^p \) is Banach space.

II. MATHEMATICAL FORMULATION

Proposition 2.1

If \( \varphi \) is in \( L^\infty(T) \), then \( H^2 \) is an invariant subspace for \( \varphi \), if and only if \( \varphi \) is in \( H^\infty \).

Proof:

Let \( M_\varphi \) is the multiplication operator defined by \( M_\varphi f = \varphi f \) for \( f \in L^2(T) \).

If \( M_\varphi H^2 \) is contained in \( H^2 \) since 1 is in \( H^2 \) and hence \( \varphi \) is in \( H^{\infty} \).

Conversely,

If \( \varphi \) is in \( H^\infty \)

Then \( \varphi \cdot \mathcal{P} \cdot \varphi \) is contained in \( H^2 \)

Since for \( P = \sum_{n=0}^{\infty} \alpha_j \chi_j \) in \( \mathcal{P}^+ \)

We have

\[
\int_0^{2\pi} (\varphi P) \chi_n \, d\varphi = \sum_{j=0}^{\infty} \alpha_j \int_0^{2\pi} \varphi \chi_{j+n} \, d\varphi = 0 \quad \text{for} \quad n > 0
\]

Since \( H^2 \) is the closure of \( \mathcal{P}^+ \), we have \( \varphi H^2 \) contained in \( H^2 \)

which completes the proof.

Corollary 2.2

The space \( H^\infty \) is an algebra.

Proof:

If \( \varphi \) and \( \psi \) are in \( H^\infty \)

Then \( M_\varphi M_\psi H^2 \subset M_\varphi H^2 \subset H^2 \)

By the above proposition

\( H^2 \) is invariant subspace of \( H^\infty \) then implies that \( \varphi \psi \) is in \( H^\infty \) Hence \( H^{\infty} \) is an algebra.

Theorem 2.3

If \( \mu \) is in the space \( M(T) \) of Borel measures on \( T \) and \( d\mu = 0 \) for \( n \) in \( \mathbb{Z} \) then \( \mu = 0 \).

Proof:

Since the linear span of the functions \( \{\chi_n\}_{n \in \mathbb{Z}} \) is uniformly dense \( C(T) \) and \( M(T) \) is the dual of \( C(T) \), the measure \( \mu \) represents the zero functional and hence must be the zero measure.

Corollary 2.4

If \( f \) is a function in \( L(T) \)

Such that \( \int_0^{2\pi} f(e^{i\theta}) \chi_n(e^{i\theta}) \, d\varphi = 0 \quad \text{for} \quad n \in \mathbb{Z} \).

\( f = 0 \) a.e.

Proof:

If we define the measure \( \mu \) on \( T \) such that
\[
\mu(E) = \int_E f(e^{i\varphi})\,d\varphi
\]

By hypothesis become \( \int_0^\infty \chi_n\,d\mu = 0 \) for \( n \) in \( \mathbb{Z} \).

Hence \( \mu = 0 \) and \( f = 0 \) a.e.

**Corollary: 2.5**

If \( f \) is a real-valued function in \( H^1 \), then \( f = \alpha \) a.e. for some \( \alpha \) in \( \mathbb{R} \).

**Proof:**

If we set \( \alpha = \left( \frac{1}{2\pi} \right) \int_0^{2\pi} f(e^{i\varphi})\,d\varphi \)

Then \( \alpha \) is real and

\[
\int_0^{2\pi} (f - \alpha)\,d\varphi = 0 \quad \text{for } n \geq 0
\]

Since \( f - \alpha \) is real valued; taking the complex conjugate of the proceeding equation yields.

\[
\int_0^{2\pi} (f - \alpha)\,d\varphi = \int_0^{2\pi} \chi_n\,d\varphi = 0 \quad \text{for } n \geq 0
\]

Combining this with previous identity yields.

\[
\int_0^{2\pi} (f - \alpha)\,d\varphi = 0 \quad \text{for all } n,
\]

Hence \( f = \alpha \) a.e.

**Corollary: 2.6**

If both \( f \) and \( \bar{f} \) are in \( H^1 \), then \( f = \alpha \) a.e. for some \( \alpha \) in \( C \).

**Proof:**

Apply previous corollary to the real valued functions

\[
\frac{1}{2}(f + \bar{f}) \quad \text{and} \quad \frac{1}{2}(f + \bar{f})/i
\]

**Given:**

\( f \) and its conjugate \( \bar{f} \) are in \( H^1 \)

\( \Rightarrow \frac{1}{2}(f + \bar{f}) \) and \( \frac{1}{2}(f + \bar{f})/i \) belong to \( H^1 \)

Let \( \alpha = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\varphi})\,d\varphi \)

Then \( \alpha \) is complex

And \( \int_0^{2\pi} \left( \frac{1}{2}(f + \bar{f})/i - \alpha \right)\chi_n\,d\varphi = 0 \quad \text{for } n \geq 0 \)

taking complex conjugate

\[
\int_0^{2\pi} \left( \frac{1}{2}(f + \bar{f})/i - \alpha \right)\overline{\chi_n}\,d\varphi = 0 \quad \text{for } n \geq 0
\]

combining above two results

\[
\frac{1}{2}(f + \bar{f}) = \alpha \quad f + \bar{f} = 2\alpha \Rightarrow 2f = 2\alpha \text{ (if } f = \bar{f})
\]

\( f = \alpha \) a.e. where \( \alpha \in C \quad f \in H^1 \)

**Borel Theorem: 2.7**

If \( \mathcal{M} \) is a positive Borel measure on \( T \), then a closed subspace \( \mathcal{M} \) of \( L^2(\mu) \) satisfies \( \chi_\alpha \mu = \mathcal{M} \) if and only if there exist a Borel subset \( E \) of \( T \) such that

\[
E = L^2(\mu) = \{ f \in L^2(\mu): f(e^{it}) = 0 \quad \text{for } e^{it} \in E \}
\]

**Proof:**

If \( \mathcal{M} = L^2(\mu) \) then clearly \( \chi_\alpha \mu = \mathcal{M} \)

Conversely, if \( \chi_\alpha \mu = \mathcal{M} \) then it follows that \( \mathcal{M} = \chi_\alpha \chi_\alpha = \chi_\alpha \) and hence \( \mathcal{M} \) is a reducing subspace for the operator \( \mathcal{M}_{\chi_\alpha} \) on \( L^2(\mu) \).

Therefore if \( F \) denotes the projection on to \( \mathcal{M} \), then \( F \) commutes with \( \mathcal{M}_{\chi_\alpha} \)

**By proposition:**

"If \( T \) is an operator on \( \mathcal{H} \), \( \mathcal{M} \) is a closed subspace of \( \mathcal{H} \) and \( P_M \) is a projection on to \( \mathcal{M} \) then \( \mathcal{M} \) is an invariant subspace for \( T \), if and only if \( P_M TP_M = TP_M \) if and only if \( \mathcal{M} \) is an invariant subspace for \( T^* \) further, \( \mathcal{M} \) is a reducing subspace for \( T \) if and only if \( P_M TP_M = TP_M \) and only if \( \mathcal{M} \) is an invariant subspace for both \( T \) and \( T^* \) with \( M_{\phi} \) for \( \phi \) in \( C(T) \)

By corollary if \( x \) is a compact Hausdroff space and \( \mu \) is a finite positive regular Borel measure on \( X \), then \( C(X) \) is \( W^* \)-dense in \( L^\infty(\mu) \).

The algebra \( m = \{ M_{\phi} \phi \in L^\infty(\mu) \} \) is maximal abelian

Now conclude that \( F \) is of the form \( M_{\phi} \) for some \( \phi \) in \( L^\infty(\mu) \)

Hence \( \mathcal{M} = \{ f \in L^2(\mu): f(e^{it}) = 0 \quad \text{for } e^{it} \in E \}

**Theorem: 2.8**

If \( \mu \) is a positive Borel measure on \( T \), then a nontrivial closed subspace \( \mathcal{M} \) of \( L^2(\mu) \) satisfies \( \chi_\alpha \mathcal{M} = \mathcal{M} \) and \( \cap_{\alpha \neq \alpha} \mathcal{M} = \{ 0 \} \)

if and only if there exist a Borel function \( \phi \) such that \( |\phi|^2\,d\mu = \frac{d\phi}{2\pi} \)

then the function \( \psi f = \phi f \) is \( \mu \) measurable for \( f \) in \( H^2 \)

and

\[
||\psi f||^2 = \int_T ||\psi f||^2\,d\mu = \frac{1}{2\pi} \int_0^{2\pi} |f|^2\,d\varphi = ||f||^2
\]

\( \varphi \) has the image \( \mathcal{M} \) of \( H^2 \) under the isometric \( \psi \) is a closed subspace \( L^2(\mu) \), \( \mu \) is invariant for \( M_{\chi_\alpha} \).

Since \( \chi_\alpha (\psi f) = \psi (\chi_\alpha f) \) then we have

\( \cap_{\alpha \neq \alpha} \mathcal{M} = \psi [\cap_{\alpha \neq \alpha} \mathcal{M}] = \{ 0 \}\)

Hence \( \mathcal{M} \) is a simply subspace for \( M_{\chi_\alpha} \) conversely suppose \( \mathcal{M} \) is a nontrivial closed invariant subspace for which \( M_{\chi_\alpha} \)

\( \cap_{\alpha \neq \alpha} \mathcal{M} = \{ 0 \} \).

Then \( L = \mathcal{M} \cap \chi_\alpha \mathcal{M} \) is non trivial and \( \mathcal{M} = \chi_\alpha \mathcal{M} \cap \chi_{\alpha+1} \mathcal{M} \)

since multiplication by \( \chi_\alpha \) is an isometry on \( L^2(\mathcal{M}) \) therefore, the subspace \( \cap_{\alpha \geq 0} \Theta \chi_{\alpha+1} \mathcal{M} \) is contained in \( \mu \) and an easy argument reveals

\( \mathcal{M} = \{ \cap_{\alpha \geq 0} \Theta \chi_{\alpha+1} \mathcal{M} \} \) to be \( \cap_{\alpha \geq 0} \mathcal{M} = \{ 0 \} \)

If \( \phi \) is a unit vector in \( L \), then \( \phi \) is n orthogonal to \( \chi_\alpha \mathcal{M} \)

hence to \( \chi_\alpha \phi, n > 0 \) and thus we have

\[
(\phi, \chi_\alpha \phi) = \int_T |\phi|^2 \chi_\alpha \,d\mu = \int_T |\phi|^2 \chi_\alpha \,d\mu
\]

We see that \( |\phi|^2\,d\mu = \frac{d\phi}{2\pi} \)

Suppose \( L \) has dimension greater than one and \( \phi^\perp \) is a unit vector in \( L \) orthogonal to \( \phi \).

In this case we have

\[
0 = (\chi_\alpha \phi, \chi_\alpha \phi^\perp) = \int_T \phi \phi^\perp \chi_\alpha \,d\mu \quad \text{for } n, m \geq 0
\]

Thus \( \int_T t_k \,d\mu = \int_T \phi^\perp \chi_{m-n} \,d\mu = 0 \quad \text{for } k \) in \( \mathbb{Z} \)
Where $dv = \varphi d\mu$. Therefore $\varphi\varphi^* d\mu = 0 \mu$ a.e combining this with that $|\varphi|^2 d\mu = |\varphi^*|^2 d\mu$ leads to a contraction and hence $L$ is one dimensional. Thus we obtain that $\varphi P_\circ$ is dense and hence $M = \varphi H^2$ hence the proof.

**Beurling Theorem:** 2.9

A function $\varphi$ in $H^\infty$ is an inner function if $|\varphi| = 1$ a.e. If $T_{\chi_1} = M_{\chi_1}|H^2$, then a nontrivial closed subspace $M$ of $H^2$ is invariant for $T_{\chi_1}$ if and only if $M = \varphi H^2$ for some inner function $P$.

**Proof:**

If $\varphi$ is an inner function, then $\varphi P_\circ$ is contained in $H^\infty$, since the later is an algebra and is therefore contained in $H^2$, since $\varphi H^2$ is the closure of $\varphi P_\circ$.

We see that $\varphi H^2$ is a closed invariant subspace for $T_{\chi_1}$ Conversely, if $M$ is a nontrivial closed invariant subspace for $T_{\chi_1}$ then $M$ satisfies the hypotheses of the proceeding theorem for $\varphi H^2$. Hence there exists a measurable function $\varphi$ such that $\mu = \varphi H^2$ and $|\varphi|^2 d\varphi / 2\pi = d\varphi / 2\pi$

Therefore $|\varphi| = 1$ a.e. since 1 is in $H^2$

We get $\varphi = \varphi_1$ is in $H^2$

Thus $\varphi$ is an inner function.

**Theorem:** 2.10

If $\mu$ is a positive Borel measure on $T$ then a closed invariant subspace $M$ for $M_{\chi_1}$ has a unique direct sum decomposition $M = M_1 \oplus M_2$ such that each $M_1$ and $M_2$ is invariant for $M_{\chi_1}$, $\chi_1 M_1 = M_1$ and $\cap_{n \geq 1} \chi_n M = \{0\}$

**Proof:**

If we set $M_2 = \cup_{n \geq 0} \chi_n M$ then $M_2$ is a closed invariant subspace for $M_{\chi_1}$ satisfying $\chi_1 M_2 = M_2$ the function $f$ in $M$ if and only if it can be written in the form $\chi_n g$ for some $g$ in $M$ for each $n > 0$.

If we set $M_2 = M - M_1$

Then function $f$ in $M$ is in $M_2$, if and only if $(f, g) = 0$ for all $g$ in $M_1$

Since $0 = (f, g) = (\chi_1 f, \chi_1 g)$ and $\chi_1 M_1 = M_1$ it follows that $\chi_1 f$ is in $M_2$ and hence $M_2$ is invariant for $M_{\chi_1}$ if $f$ is in $\cap_{n \geq 1} \chi_n M_2$ then it is in $M_1$ and hence $f = 0$ hence the proof.

**Theorem:** 1.5

If $f$ is a nonzero function in $H^2$, then the set $\{e^{it} \in T : f(e^{it}) = 0\}$ has measure zero.

**Proof:**

Set $E = \{e^{it} \in T : f(e^{it}) = 0\}$ and define $M = \{g \in H^2 : g(e^{it}) = 0 \text{ for } e^{it} \in E\}$

it is clear that $M$ is a closed invariant subspace for $T_{\chi_1}$ which is nontrivial since $f$ is in it.

By beurling theorem there exist inner function $\varphi$ where $|\varphi| = 1$ a.e. Such that $M = \varphi H^2$ since 1 is in $H^2$ it follows that $\varphi$ is in $M$ and hence that $E$ is contained in $\{e^{it} \in T : \varphi(e^{it}) = 0\}$. Since $|\varphi| = 1$ a.e. Hence the measure zero.

**III. Conclusion**

The theory of Hardy spaces and its related theorems are discussed. The invariant subspace of the borel measures for hardy spaces are also proved.

**References**